THOMSON'S NEW MATHEMATICAL SERIES.

THE

COLLEGIATE ALGEBRA;

ADAPTED TO

COLLEGES AND UNIVERSITIES.

BY

JAMES B. THOMSON, LL. D.,

AUTHOR OF NEW MATHEMATICAL SERIES,

AND

ELIHU T. QUIMBY, A. M.,

LATE PROFESSOR OF MATHEMATICS IN DARTMOUTH COLLEGE.

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PREFACE.

SOON after the publication of the "New Practical Algebra," the author was urgently requested by several mathematical professors to prepare a higher work on the same general plan, adapted to the wants of Colleges and Universities. In compliance with these requests, the present treatise was undertaken and is now presented to the public.

To facilitate its preparation he was fortunate in securing the co-operation of Prof. E. T. Quimby, of Dartmouth College, a gentleman of more than twenty-five years experience in teaching mathematics.

The work presents a full discussion of all the subjects usually contained in the most complete text-books in use, as the Demonstration of the Binomial Formula, the Computation of Logarithms, Theory of Equations, Sturm's Theorem, Indeterminate Coefficients, Series, Infinitesimal Analysis, Horner's Method of Approximation, Loci of Equations, Exponential Equations, etc.

A few subjects, as Probabilities, etc., thought to be of less importance have been thrown into an Appendix. While the most approved authors have been freely consulted and their methods carefully compared, the plan and execution of the work are the results of long personal experience in the class-room.

The arrangement is systematic, each subject appearing in its natural order, and the dependence of the principles upon each other is shown by frequent references.

The Examples are numerous, and have been prepared with a view both to illustrate the principles under discussion, and to stimulate thought on the part of the student.

Great pains have been taken to make the rules and definitions clear and concise, and the demonstrations simple, rigorous, and logical. The subject has been brought down to the present time, and the best of the improved methods of teaching the various topics have been adopted, to the exclusion of such as are obsolete.

Originality of *matter* is not to be expected in a book of this kind; but it will be found that several subjects have been treated in a *manner* more or less original. The reader is referred to the Articles on the use of the Directive Signs, or Factors of Direction, the treatment of Imaginary Quantities, Logarithms, Series, etc.

The work is designed to meet a want which has been felt to a greater or less extent, but which heretofore has not been supplied in a satisfactory manner.

It is earnestly commended to the attention of instructors and students, with the hope that it will be found to contain enough that is new and useful to satisfy, in a measure, the views of those who believe in progress, and who have desired some departures from the beaten track.

In conclusion, the authors would avail themselves of the opportunity to express their obligations to their friends who have favored them with many valuable suggestions upon the subject.

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ALGEBRA.

INTRODUCTION.

Art. 1. ***Mathematics*** is the *science of quantity.*

2. ***Quantity*** is anything which *can be measured;* as, *distance, space, time,* etc.

3. The ***Measure of a Quantity*** is the *number of times* it contains another quantity of the same kind, called the *unit of measure;* or, it is the *ratio* of *the quantity* to the *unit of measure.* Hence,

4. ***Number*** is the *measure of quantity.*

5. A quantity is measured *mechanically* by applying the unit of measure *directly* to the quantity, and counting the *number* of times it is applied.

Thus, the application of the yard-stick to measure the *length* of a piece of cloth, and of the surveyor's chain to measure *distances,* are examples of the *mechanical measurement* of quantities.

6. When the *unit* is not contained in the quantity to be measured an *integral* number of times, this unit may be divided into any number of equal parts, and *one of these parts* taken as a unit. In this way, *a fraction* or *a mixed number* may express the measure of a quantity.

Thus, we may have a piece of cloth $5\frac{1}{2}$ or $5\frac{3}{4}$ yards long. The exact measure of a quantity cannot be found with a unit which cannot be divided into such a number of equal parts that one of these parts shall be contained an *integral* number of times in the quantity to be measured.

7. ***Commensurable Quantities*** are those that *can be measured with the same unit.*

8. ***Incommensurable Quantities*** are those that *cannot be measured with the same unit.*

Thus, the *side* of a square and its *diagonal* are distances that *cannot be measured with the same unit.* However short the unit may be with which we attempt to measure these two lines, if it give an exact measure of one, it will not of the other.

9. A ***Single Quantity*** is called *commensurable* or *incommensurable* according as it *can* or *cannot* be measured with the unit we are using.

10. ***Quantities*** which are of *different kinds or natures*, as *time* and *distance*, cannot be compared one with the other; for, the *magnitude* or *extent* of one is wholly *unlike* that of the other, so that one cannot be made the unit of measure for the other.

11. ***Any quantity*** may be made the ***unit of measure*** for quantities of its ***own kind***, but for the *purposes of trade* and other mutual uses, the unit must be generally known and accepted; hence the necessity of units established by law as a national standard.

NOTE.—In common language, we speak of *measuring* a *pile* of *wood*, or a *piece* of *land*, but, strictly speaking, that which we ***really measure*** is the *space* occupied by the wood, and the *area* of the land. So also we *measure* the *length*, the *weight*, the *density*, the *elasticity*, etc., of material bodies, but not the *bodies* themselves.

12. Quantities as they appear in nature have certain definite relations to each other, which make them mutually dependent, and from which the measure of one may be found when the measures of others are known.

Mathematics *investigates* these relations and *determines* the measures of quantities *indirectly*, or without *direct measurement.*

13. Quantities thus mutually dependent are called ***Functions*** of each other. For example,

14. In all motion, three quantities are involved, viz.: ***Time, Distance,*** and ***Velocity.*** These quantities are so related that neither can change without changing one or both of the others. If the velocity increase, the distance will increase or the time decrease, or both these results may follow.

NOTES.—1. In common language, we say the *time depends* on the *distance* and *velocity;* the *distance* on the *time* and *velocity;* and the *velocity* on the *time* and *distance.*

2. In *mathematical language,* the *time* is a ***function*** of the *distance* and *velocity;* the *distance* is a ***function*** of the *time* and *velocity;* and the *velocity* is a ***function*** of the *time* and *distance.*

15. The question, "*What function* is one quantity of others?" refers to the manner in which the latter quantities must be combined or treated to give the *measure* of the former; as, "*What function* is the *distance* of the *time* and *velocity?*" The answer to which would be, "*The product*"; that is, the distance is the *product* of the time by the velocity.

16. In like manner, let the pupil answer the following questions:

1. What function of the *side* of a square is its *area?*
2. What function of the *radius* of a circle is its *diameter?*
3. What function of the *diameter* of a circle is its *circumference?* Its *area?*
4. What function of the *principal, rate per cent,* and *time,* is the *interest* on a note? The *amount?*
5. What function of the *number of pounds* and the *price per pound* is the *cost* of an article?
6. What function of its *sides* is the *area* of a rectangle?

17. A ***Proposition*** is *something proposed* for *demonstration,* or *solution.*

18. A ***Theorem*** is a proposition *for demonstration.*

19. A ***Problem*** is a proposition for *solution.*

NOTE.—A *theorem affirms,* "*This is true,*" and requires *demonstration.*

A *problem inquires,* "*What is true?*" and requires *solution.*

20. An ***Axiom*** is a *self-evident theorem.*

21. A ***Postulate*** is a *self-evident problem.*

NOTE.—A truth is called *self-evident* when it commands the instant assent of one who is acquainted with the subject to which it relates, and cannot be made plainer by any proof.

22. A ***Demonstration*** is an arrangement of *definitions, axioms,* and *postulates,* by which the truth of a *theorem* is established.

NOTE.—A *direct* demonstration proves that a theorem *is true*, by assuming the truth of certain definitions and axioms, and from these premises deducing other truths, till we arrive at the one which is to be established.

An *indirect* demonstration proves that a theorem is ***not untrue***, by proving that the supposition of its *contrary* involves an absurdity.

23. A ***Solution*** is an arrangement of *axioms* and *postulates* by which the answer to a *problem* is determined.

24. The ***Hypothesis*** of a *proposition* is:

1st. In a theorem;—*The conditions* on which the theorem is affirmed.

2d. In a problem;—*The data* from which the required truth is to be determined.

25. A ***Corollary*** is *an inference* from a preceding *demonstration* or *solution.*

26. An ***Equation*** is *an expression of equality* between two quantities.

The equation is used to express in *algebraic language* the *relations between quantities* which are functions of each other.

27. ***Known Quantities*** are those from which other quantities are to be determined. *Arbitrary values* may therefore be assigned to them at pleasure.

28. ***Unknown Quantities*** are those whose values *are to be determined* from their relations to other quantities. They are regarded as *functions* of *known quantities* and *cannot* therefore *have arbitrary values* assigned to them.

29. Problems are of two kinds:

1st. Those which require some *geometrical* or *mechanical construction;* as, To *construct* a *triangle from three given lines.*

2d. Those which require the *measure of a quantity* from its relations to *other quantities;* as, To find the *base* of a right-angled triangle *from the other sides.*

30. The ***Solution*** of a problem of the second kind is made up of *three* distinct *parts* or *steps:*

1st. *Finding the equations* which express the relations between the quantities involved.

2d. *Finding* from these equations *what function* the *unknown* quantity is of the *known* quantities.

3d. *Substituting* in this function *the numbers* representing the *known quantities.*

31. ***The first of these steps*** requires a knowledge of that branch of *mathematics* or *physics* to which the problem belongs; as, *Geometry, Mechanics,* etc.

The second is the province of *Algebra.*

The third belongs exclusively to *Arithmetic.*

32. For illustration take the following problem:

A rope 50 feet long attached to the top of a vertical pole, reaches the ground 40 feet from the foot of the pole, on a horizontal plane; how high is the pole?

First step: *By geometry* we learn that the square of the length of the rope equals the *sum* of the *squares* of the length of the pole and the distance on the ground. This relation expressed by an equation is

$$a^2 = b^2 + x^2,$$

in which the letters a, b, and x have been put for the length of the rope, distance on the ground, and height of the pole, respectively.

Second step: *By algebra* this *equation* is *reduced* to the form

$$x = \sqrt{a^2 - b^2},$$

which shows *how the known quantities must be combined* to produce the *unknown;* in other words, *what function* the *unknown* is of the *known.*

Third Step: *By arithmetic* the numbers 50 and 40 are substituted for a and b; thus,

$$x = \sqrt{50^2 - 40^2} = 30, \textit{ Ans.}$$

33. The equation $a^2 = b^2 + x^2$ cannot be true unless the quantities a, b, and x are *mutually dependent;* that is, unless *each* is *a function* of the *other two.* The equation therefore ***implies*** that x is *a function* of a and b, but does not state ***explicitly*** *what function.*

In such an equation x is said to be an ***Implicit Function*** of a and b.

The equation

$$x = \sqrt{a^2 - b^2}$$

states ***explicitly*** *what function* x is of a and b, and it is therefore called an ***Explicit Function.***

34. This change from an *implicit* to an *explicit function* is called "*reducing the equation.*"

The *explicit function* resulting from the reduction of the equation is called a ***Formula,*** and is the expression in *algebraic language* of an *arithmetical rule.*

The above *formula* for finding the *perpendicular* of a right-angled triangle when the *hypothenuse* and *base* are given, being translated into common language, becomes the following,

RULE.—*Subtract the square of the base from the square of the hypothenuse, and take the square root of the difference.*

From the preceding illustrations we have the following definitions:

35. ***Algebra*** is the *science* of the *Equation.* (Art. 34.)

Its object is the reduction of equations, by which formulas for arithmetical computations are obtained.

36. ***Arithmetic*** is the *Science of Numbers.* Its object is the *substitution* of *numbers* in *algebraic formulas;* or, what is equivalent to this, the *combination* of *numbers* in accordance with *rules* furnished by Algebra.

37. The ***Reduction of an Equation*** consists in such transformations as will make the *unknown quantity* an *explicit function* of the *known quantities.*

38. The ***Reduction of Equations*** is based on the following

AXIOMS.

1°. *Equal quantities equally affected remain equal.*

2°. *Equal quantities unequally affected become unequal.*

3°. *Unequal quantities equally affected remain unequal.*

4°. *Quantities equal to the same quantity are equal to each other.*

5°. *Quantities differing equally, in both magnitude and direction, from the same quantity are equal to each other.*

6°. *The whole is greater than its part, and is equal to the sum of all its parts.*

CHAPTER I.

NOTATION.

39. ***Notation*** in Algebra is the method of expressing quantities, their relations, and combinations, by *general symbols.* The algebraic symbols differ from the Arabic figures, or *numerical measures,* in this respect; the latter represent *specific quantities,* the former *general quantities.*

NOTE.—This notation constitutes what is called the *algebraic language.* It is necessarily *general,* since its object is to furnish *general formulas* for *arithmetical computations.*

40. The ***Symbols*** used may be classified as follows: 1st, Symbols of Quantity; 2d, of Operation; 3d, of Relation; 4th, of Abbreviation.

SYMBOLS OF QUANTITY.

41. ***Quantities are commonly represented*** by the *letters* of the alphabet. *Any letter* may be used to represent *any quantity,* and the same letter may represent *different* quantities, subject to one limitation; the *same* letter must always stand for the *same* quantity throughout the *same* discussion.

42. For uniformity and convenience, the following order should be observed:

1st. It is customary to employ the *first letters* of the alphabet to represent *known quantities,* as, a, b, c, etc.; and the *last* for *unknown quantities,* as, x, y, z, etc.

2d. Initial letters are frequently used; as, r or R for *radius;* c for *circumference;* s for *sum,* etc.

3d. ***Different Quantities*** of the same kind may be represented, *in the same problem*, by the same letter, with accents or subscript figures to distinguish the different quantities.

Thus, when a problem involves the radii of several circles, we may use r', r'', r''', etc. (read, "r prime," "r second," "r third," etc.), or r_1, r_2, r_3, etc. (read, "r sub one," "r sub two," etc.).

4th. The Greek letters are commonly used to represent angles, but sometimes other quantities.

Thus, the *Greek* π is used for the *ratio* of the *circumference* of a circle to its *diameter*.

5th. The symbol ∞ (the figure 8 placed horizontally) represents *infinity*, or a quantity *greater than any assignable quantity.*

6th. The symbol 0 (*zero*) represents an *infinitesimal* quantity, or a quantity *less than any assignable quantity.*

43. *Quantities* when expressed by *numbers* are called ***numerical;*** when expressed by *letters*, they are called ***literal quantities.***

SYMBOLS OF OPERATION.

44. The ***Fundamental Operations*** in Algebra are ***Addition, Subtraction, Multiplication, Division, Involution,*** and ***Evolution.***

45. ***Addition*** and ***Subtraction*** are expressed as in Arithmetic by the signs + and − ; as, $a + b - c$, read, "a plus b minus c."

Quantities connected by the signs + and − are called ***Terms.***

46. When *the same term* is to be *added* or *subtracted* more than once, a number is placed before it to show how many times it is to be used.

Thus, $a+b+b-c-c-c$ may be written $a+2b-3c$.

A number thus used to show *how many times a quantity is taken as a term* is called a ***Coefficient.***

47. A ***Coefficient*** may be *integral* or *fractional,* the latter showing what *part* of a *quantity* is taken *as a term;* as, $3a$, $\frac{1}{2}ab$. It may also be *numerical,* or *literal,* or both.

Thus, in the expression $2ab$, $2a$ may be regarded as indicating how *many times* b is taken; or 2 how many times ab is taken, or b how many times $2a$ is taken.

NOTE.—The word *coefficient,* however, usually refers to the numerical factor of a term. Hence, generally,

48. A ***Coefficient*** is the *factor* or *factors* of a term indicating the *number of times* the rest of the term is taken, or, *what equal terms* are taken.

When no coefficient is expressed, 1 is always understood.

49. The *double sign* $\pm$ is used when a quantity may be either *added or subtracted,* and is read, "*plus* or *minus.*"

Thus, $a \pm b$ means that the conditions of the problem will be satisfied either by *adding* or *subtracting* b.

50. A ***Positive Quantity*** is one whose sign is $+$.

51. A ***Negative Quantity*** is one whose sign is $-$.

NOTE.—The signs $+$ and $-$ in Algebra have a more general meaning than merely *addition* and *subtraction,* which will be explained in the proper place (Arts. 81–94).

MULTIPLICATION AND DIVISION.

52. Quantities used as *multipliers* or *divisors* are called ***Factors,*** in distinction from ***Terms,*** which are *added* or *subtracted.*

53. ***Multiplication*** is expressed:

1st. By the usual sign, $\times$; as, $2 \times a \times b \times c$.

2d. By the period; as, $2 \cdot 4 \cdot 6 = 2 \times 4 \times 6$.

3d. By writing the factors one after the other without any sign; as, $2abc = 2 \times a \times b \times c$.

NOTE.—It is customary to write numerical factors first, and literal factors after, in alphabetical order, as, $3abcx$.

54. ***Division*** is expressed:

1st. By the usual sign, $\div$; as, $a \div b$.

2d. By writing the divisor under the dividend in the form of a fraction; as, $\frac{a}{b} = a \div b$.

3d. By a colon **:** ; as, $a : b = a \div b$.

4th. By a negative exponent; as, $ab^{-1} = a \div b$, as seen below.

55. When the *same factor* is used more than once, either as a *multiplier* or *divisor*, a figure called an ***Exponent*** is placed a little above and to the right of it, to show how many times it is used.

Thus, instead of *aaabb* we write a^3b^2, and for $\frac{aaa}{bb}$ we write $\frac{a^3}{b^2}$, or a^3b^{-2} (Art. 54, 4th).

INVOLUTION AND EVOLUTION.

56. ***Involution*** is the *multiplication of equal factors.*

57. ***Evolution*** is the process by which a quantity is separated into *equal factors.*

58. A ***Power*** is the product of any number of the equal factors of a quantity, and is expressed by an exponent; as, a^2, a^3, $a^{\frac{2}{3}}$, etc., read, "*a* second power" or "*a* square," "*a* third power" or "*a* cube," "*a* two-thirds power," etc.

NOTE.—In reading *powers*, do not omit the word "power," reading "*a* fourth," "*a* third," etc., for that means a'''', a''', etc.

59. A ***Root*** is one of the *equal* factors of a quantity.

It is therefore a power whose exponent is a fraction with 1 for a numerator; as, $a^{\frac{1}{2}}$, $a^{\frac{1}{3}}$, which may be read, "the square root of *a*," "the cube root of *a*," or "*a* one-half power," "*a* one-third power."

60. The ***Denominator*** of a fractional exponent shows the *number* of *equal factors* into which the quantity is separated.

61. The ***Numerator*** shows *how many* of those factors are taken. Hence,

62. An ***Exponent*** shows *what equal factors* are taken.

63. The ***Radical Sign,*** $\sqrt{\ }$, is often used to express *roots,* a figure being written over it to indicate what root is taken.

Thus, $\sqrt[3]{a} = a^{\frac{1}{3}}$; $\sqrt[4]{a} = a^{\frac{1}{4}}$; $\sqrt[3]{a^2} = a^{\frac{2}{3}}$. When no figure is written over the sign, 2 is understood; as, $\sqrt{a} = \sqrt[2]{a} = a^{\frac{1}{2}}$.

NOTE.—The figure placed over the radical sign is called the *Index* of the root, because it denotes the name of the root.

SYMBOLS OF RELATION.

64. The ***Sign of Equality*** is $=$; as, $a = b + c$, read, "a equals $b + c$."

65. ***Inequality*** is expressed by two lines forming an acute angle, and opening towards the greater quantity; as, $a > b$, or $a < b$, read, "*a is greater than b,*" or "*a is less than b.*"

66. The ***Sign of Variation*** is $\propto$. It shows that the quantities between which it is placed have a *constant ratio.*

Thus, $x \propto y$, read, "x varies as y," means that however x may change its value, y also changes, so that the *quotient* $x \div y$ *does not change.*

SYMBOLS OF ABBREVIATION.

67. The symbol $\therefore$ is used for the word *therefore,* and $\because$ for *because.*

68. The ***Vinculum,*** horizontal, $\overline{\quad\quad}$, or vertical, $|$, the ***Parenthesis,*** ( ), ***Brackets,*** [ ], and ***Braces,*** { }, are used to connect several quantities with the same *coefficient, exponent,* or *sign.*

Thus, $\left\{\left[(a + b - c)^2 - \overline{x + y + z}^2\right]^3 - (x - a)\right\}^{\frac{1}{2}}$, indicates

1st. That the quantity $a+b-c$ is to be *squared.*

2d. That the quantity $x+y+z$ is to be *squared.*

3d. That the second of these squares is to be subtracted from the first, and the difference raised to the third power.

4th. That the quantity $(x-a)$ is to be subtracted from this cube, and the square root of the difference taken.

It is sometimes convenient to use the *vinculum* vertically; as, $\left.\begin{matrix} a \\ +b \\ -c \end{matrix}\right| x$ which is the same as $(a+b-c)x$.

69. In writing a *series of terms,* or *factors,* the expression is often abbreviated by omitting a part of the quantities, where they can be easily supplied, and indicating the omission by a succession of *dots* or *short dashes.*

Thus, $1+2+3 \ldots . 8$; meaning the sum of the numbers 1, 2, 3, to 8, inclusive.

The product of the numbers 1, 2, 3, etc., to any given number may be expressed in this manner $(1 \cdot 2 \cdot 3 \ldots . 10)$, but it is usually still further abbreviated by writing the last factor; thus,

$$\underline{|10} = 1 \cdot 2 \cdot 3 \ldots . 10.$$

This is read, "*factorial* 10," meaning the product of the natural numbers from 1 to 10 inclusive. So

$$\underline{|n} \text{ (read "factorial } n\text{")} = 1 \cdot 2 \cdot 3 \ldots . n.$$

ALGEBRAIC EXPRESSIONS.

70. An ***Algebraic Expression*** is any quantity expressed in algebraic language; as, $3a$, $5a - 7b$, etc.

71. The ***Terms*** of an algebraic expression are the quantities which are connected by the signs $+$ and $-$.

Thus, in $a + b$ there are two terms; in $x + y \times z - a$ there are three, $y \times z$ being a single term. For, quantities connected by the signs $\times$ or $\div$ do not constitute separate terms.

72. A ***Monomial*** is an algebraic expression containing only *one term;* as, a, $2ab$, etc.

73. A ***Binomial*** has *two terms;* as, $a + b$.

74. A ***Trinomial*** has *three terms;* as, $a + b + c$.

75. A ***Polynomial*** has *three* or *more terms;* as,

$$ax + by - z - x.$$

NOTE.—A *binomial* is sometimes called a *polynomial.*

76. The ***Degree*** of a term is the number of its *literal factors.* As the number of these factors is indicated by the *exponents,* the degree of a term will be the *sum of the exponents of its literal factors.*

Thus, $2ab$, $2a^2$, and $3ax$ are of the *second* degree, and a^2b, ab^2, and b^3 are of the *third* degree.

77. A ***Homogeneous Polynomial*** has all its terms of the same degree.

Thus, $ab^2 + a^2b + b^3$ is homogeneous, but $a^2b^2 - 2ab + b^2$ is not homogeneous.

78. ***Like*** or ***Similar Terms*** are those containing the *same powers* of the *same letters;* as, a^2x and $2a^2x$.

79. ***Unlike*** or ***Dissimilar Terms*** contain different *letters,* or the same letters with different *exponents.*

Thus, $2a^2x$ and $2ax^2$, $2a^2$ and $2a^2x$, a^2x^2 and a^2x^2y, etc., are dissimilar terms.

80. The ***Reciprocal*** of a quantity is *unity divided by that quantity.*

Thus, a and $\frac{1}{a}$ are *reciprocals* of each other.

POSITIVE AND NEGATIVE QUANTITIES.

81. The *signs* $+$ and $-$ are used to indicate *addition* and *subtraction;* but if we inquire why certain quantities in the solution of a problem are *added* and others *subtracted,* we shall find that it depends on the *direction* of the quantities. This will be best illustrated by a few examples.

1st. A man walks several distances, some *north* and some *south.* How far *north* of his starting-point does he stop?

To answer this, we *add north distances* and *subtract south,* because the former *increase,* while the latter *decrease* the result.

How far *south* of his starting-point does he stop?

Here we *add south distances* and *subtract north,* for the same reason.

2d. A man has *bills payable* and *bills receivable.* What is their *net value* to him?

Add *bills receivable* and subtract *bills payable.* They have *opposite directions;* one represents cash *coming to* him and the other *going from* him.

What is the net amount of *debt* these bills represent?

Add *bills payable* and subtract *bills receivable,* the question having been reversed.

3d. What is the *value* of my bank account?

Add *deposits,* subtract *drafts,* because the former *come to me,* the latter *go from me; opposite directions* as before.

We might also give illustrations involving *time past* and *future,* suggestive of direction *backwards* and *forwards,* as well as other varieties of oppositeness, *not* in the *nature of the quantity, but in its relation to the problem;* all of which, without severe stretch of the imagination, may be called *opposite directions.*

82. These *opposite directions* determine whether a quantity shall be *added* or *subtracted;* that is, they determine *the direction* in which the quantity is to be used; for *addition* and *subtraction,* by which a quantity is *put in* or *taken out* as a term, may be considered as *opposite directions.*

83. It will be observed that a quantity having a particular direction is not always to be *added,* nor is one having the opposite direction always to be *subtracted;* but when the conditions require one to be *added,* the other must be *subtracted.*

84. These *opposite directions* are called ***Positive*** and ***Negative;*** that direction which tends to *increase* the quantity sought being usually called *positive,* and the opposite direction *negative,* though either direction may be assumed as *positive* at pleasure.

85. Since the signs + and — represent the *direction* of a quantity, they may be called ***Directive Signs,*** or ***Factors of Direction.***

86. In the use of these signs it is necessary to consider the following

PRINCIPLES.

1°. A *quantity may be considered without reference to its direction.* It has then *no sign,* and is neither *positive* nor *negative.*

Thus, the answers to the questions, How far? How long? How many? etc., are of this nature.

But when a question includes the idea of direction, as, How far *north?* How long *before?* How long *after?* How much did you *receive?* etc., the answer must be either + or —.

2°. *Questions involving the idea of direction may always be reversed.*

Thus, How far *north?* How far *south?* How much did you *receive?* How much did you *give?* etc.

3°. *The nature of problems is such, that the quantities involved must have one of two opposite directions.*

When a problem asks, "How far *north* or *south?*" other directions have nothing to do with the answer. So of *time past* or *future, bills payable* and *bills receivable,* etc., the only *possible directions* are *two,* and these are *opposite.*

NOTE.—Nothing but some *impossible condition* in a problem will introduce into the solution a quantity out of the line of *positive* and *negative,* and when by reason of an impossible condition such a quantity is introduced, it is called an *impossible* or *imaginary quantity,* and its direction is expressed by a method to be explained hereafter (Art. 292.)

4°. *There is no direction which is naturally positive,* but it is customary to consider the *direction named in a problem* as *positive,* unless the *opposite* be made so by special assumption.

87. In operations upon quantities having *directive signs,* it is not necessary to consider the *nature* of the *quantities,* nor the *kind of oppositeness* which gave rise to these signs, whether of *distance* or *time,* or of *debt* and *credit;* for when once the equations are formed, the signs are used in accordance with the arbitrary meaning assigned to them; that is, in accordance with the following *definitions* of the *directive signs.*

DIRECTIVE SIGNS.

88. The *sign* + indicates the *positive direction*, but has *no power* to control the direction of a quantity in the presence of the sign —.

89. The *sign* — *reverses* the *direction* of a quantity.

90. In the application of these definitions we observe:

1st. The sign + having *no power* as a *factor of direction*, may be omitted, except when its omission would lead to some misunderstanding; just as the factor 1 is omitted because it has *no power* as a *factor of magnitude* or *value*.

2d. Two or more + signs mean no more than one, as two or more 1's as factors have no more effect than one such factor.

3d. When a quantity has several signs, some of which are + and some —, the direction of the quantity will depend wholly on the *negative signs*.

4th. When a quantity has several minus signs, each of these signs will *reverse* its direction. For illustration:

Let the minute-hand of a clock, when pointing to the hour XII, represent the positive direction of a quantity, or its direction with *no written sign*.

A single *minus* sign will *reverse that direction*, turning the hand *backwards** till it points to VI.

Another minus sign will continue this revolution, and bring the hand back to XII, or the positive direction. Thus we see that $-a$ is negative, $--a$ is positive, and $---a$ is negative, etc. These may therefore be written $-a$, $+a$, $-a$, etc., or if we wish to show how many minus signs are used in giving the direction, we may use an *exponent* for this *factor of direction*, as well as for *factors of magnitude*, and write, $-a$, $-^2a$, $-^3a$, $-^4a$, etc.

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* We say *backwards* (meaning opposite to the natural motion of the hands of the clock), because it is customary to consider revolution in this direction as *positive*, and in the opposite direction *negative*. Taking away a — sign from a quantity would reverse it in the negative direction. There is nothing, however, to forbid reversing this supposition and making *positive* revolution agree with the natural motion of the hands.

91. From these illustrations we have the following

Rule.—*An even power of* $-$ *is positive, an odd power negative.*

Or, *An even number of minus signs gives* $+$, *and an odd number* $-$.

92. Apply this rule to the following

EXAMPLES.

1. What is the sign of $+ - + a$?

2. What is the sign of $-^2 - + -^2 a$?

3. What is the sign of a and what of b in the following expression: $- + (a - b)$?

4. What is the sign of $- \pm + a$? *Ans.* $\mp$.

5. What is the sign of $- - \pm a$? *Ans.* $\pm$.

6. What is the sign of $\pm - \mp a$? *Ans.* $+$.

Note.—When there are several double signs, the upper signs are generally taken together, and also the lower. Thus, in the last example, the sign of a will be either $+ - -$ or $- - +$, each of which will give $+$.

93. What signs should be given to the following answers?

7. How far north did you go? *Ans.* 10 miles.

8. How much did you pay? *Ans.* 5 dollars.

9. How much older are you than John? *Ans.* 5 years younger.

10. How much older than Henry? *Ans.* 3 years.

11. How many books have you? *Ans.* 10.

12. How many books did you take from the table? *Ans.* 5.

13. How old are you? *Ans.* 50 years.

14. How far is it to New York? *Ans.* 100 miles.

Some of the above answers have no sign; which are they, and why?

94. The force of a directive sign is limited to the *term immediately following,* unless several terms are connected by a parenthesis or vinculum.

Thus, in $-a^2b+ab^2$, the sign $-$ affects only a^2b; but if we write $-(a^2b+ab^2)$, the whole quantity is negative.

95. In like manner, the sign $-$ before a fraction; as, $-\frac{a-b}{2}$, affects the whole fraction, and if in the course of a solution the denominator be removed, it must not be written $-a-b$, but $-(a-b)$, or $-a+b$.

EXERCISES IN NOTATION.

96. To Translate an Algebraic Statement from Common into Algebraic Language.

1. The *product* of the *sum* and *difference* of any two quantities is equal to the *difference of their squares.*

SOLUTION:—In this translation, it is necessary first to assume letters to represent the quantities. Let a and b be the letters. Then the statement becomes $(a+b)(a-b)=a^2-b^2$, *Ans.* Hence, the

RULE.—*For the words, substitute the letters and signs which indicate the relations of the quantities and the operations to be performed.*

Translate the following into algebraic language:

2. The square of the sum of any two quantities is equal to the sum of their squares increased by twice their product.

3. The square of the difference of any two quantities is equal to the sum of their squares decreased by twice their product.

4. The square of the sum of any two quantities added to the square of their difference is equal to twice the sum of their squares.

5. The difference of the squares of two quantities divided by the difference of the quantities equals the sum of the quantities.

6. The difference of the square roots of two quantities divided by the sum of their fourth roots equals the difference of their fourth roots.

7. The square of the difference of two quantities subtracted from the square of their sum is equal to four times their product.

97. To Translate Algebraic into Common Language.

1. Translate $(x+y)^2-(x-y)^2=4xy$ into common language.

SOLUTION.—The square of the sum of any two quantities, diminished by the square of their difference, is equal to 4 times their product. Hence, the

RULE.—*For the letters representing quantities and the signs indicating the given relations and operations, substitute words.*

Translate into *common language* the following

2. $(a+b)^3=a^3+3a^2b+3ab^2+b^3$.

3. $\frac{1}{2}(a+b)+\frac{1}{2}(a-b)=a$.

4. $\frac{1}{2}(a+b)-\frac{1}{2}(a-b)=b$.

98. If $a=3$, $b=2$, $c=\frac{1}{2}$, $m=a+b$, $n=a-b$, what are the numerical values of the following expressions:

5. $\dfrac{\frac{1}{2}(a-b)(a+b)}{c^2}$.

6. $2a^2+b^2-2(a^2-b^2)-2(mn-1)$.

7. $\left(a^2-b^2-\dfrac{a-b+1}{2}\right)^{\frac{1}{2}}$.

8. $\sqrt{c^2[(a+b)^2-a^2]}$.

9. $\overline{a^2+b^2-(a-b)^2}^2$.

10. $\dfrac{5\left(\dfrac{a+b}{n}-\dfrac{a-b}{m}\right)\left(abc-\dfrac{mn}{a+b}\right)}{a^3b^3c^2-(m^2+n^2)b^2c+b^2c^2}$.

NOTE.—Let the teacher add examples until the student becomes familiar with algebraic language.

CHAPTER II.

ADDITION.

99. ***Algebraic Addition*** is uniting the *terms* of two or more quantities in one expression, and *reducing* that expression to the simplest form.

100. The *result* is called the ***Sum*** or ***Amount.***

101. Algebraic addition depends upon the following

PRINCIPLES.

1°. *Similar terms only may be united in one.* (Art. 78.)

2°. *Dissimilar terms can only be connected by their signs.* (Art. 79.)

102. *Algebraic addition* differs from *Arithmetical* in the fact that the quantities added may be either *positive* or *negative.* In *Arithmetic,* the signs + and − are used merely to express addition and subtraction, and quantities, whether added or subtracted, are used as positive.

103. In Algebra, for convenience, we speak of finding the aggregate of terms of both kinds, positive and negative.

A man, for example, rows a boat up stream while the current floats it down. The first hour he rows 3 miles and is floated down 1 mile; the second hour he rows 4 miles and is floated down 2 miles; the third hour he rows 2 miles and is floated down 2 miles.

Now to find the *aggregate effect* of the oars and current is a problem for *Algebraic Addition.*

In *Arithmetic*, it would be called in part *subtraction*, as it really is, and the process of aggregating the quantities in Algebra does not differ from that which would be employed in Arithmetic.

Calling the distances the man rows *positive*, the distances he is floated, being in the opposite direction, would be *negative*, and the aggregate effect is evidently the *difference* of the positive and negative distances, which in this case is $9 - 5$, or 4 miles.

104. The relation, however, of rowing and current might be such as to make the result greater or less than this. The current might even be more rapid than the rowing, and the boat be found farther down stream than when it started.

This would be shown by the *negative* distances being greater than the *positive*, giving a negative *sum*.

NOTE.—This illustration shows the meaning of the remark, "A *negative* quantity is *less* than *zero*." Not that a negative distance is longer or shorter than it would be if positive, but when a man wishes to go up stream, it is worse than nothing to him to be floated down.

105. Hence we see that

Adding a negative quantity is equivalent to subtracting a positive one of the same numerical value, and vice versa.

Adding the effect of the current is the same in its result as taking away an equivalent rowing effect.

106. The ***Algebraic Sum*** of *several quantities* is the *difference* between the *sum of the positive* and the *sum of the negative quantities*, with the sign of the *greater sum*. Hence,

107. For *Algebraic Addition* we have the following

GENERAL RULE.

I. *Unite similar terms by prefixing the algebraic sum of the coefficients to the common letters.* (Art. 46.)

II. *Connect dissimilar terms by their signs.* (Art. 45.)

108. Terms that are similar in respect to one or more letters may be united with a polynomial coefficient.

Unite $2abc + 5abx - 3aby + 2abn$ with a polynomial coefficient.

SOLUTION.—The common letters are ab. The sum of the coefficients is $2c+5x-3y+2n$, and we have by the rule,

$$(2c+5x-3y+2n)\,ab.$$

109. In such cases, the *coefficient* of each term must be regarded as containing all the factors of the term except the *common* letters.

These coefficients therefore will sometimes be literal, in which case their sum can be expressed only by a polynomial as above. Hence it will be a matter for consideration whether the simplified expression will be more convenient for use than the full form.

EXAMPLES.

1. Add $2ab + 5cd - 8d$; $3cd - ab + 2d$; $4ax + 5ab + 2d$; $4d - 3bc - 2ax$.

SOLUTION.—For convenience, we write similar terms under each other; thus,

$$\begin{array}{rrrrr}
2ab & +\,5cd & -\,8d & +\,4ax & -\,3bc \\
-\,ab & +\,3cd & +\,2d & -\,2ax & \\
+\,5ab & & +\,2d & & \\
 & & +\,4d & & \\
\hline
6ab & +\,8cd & & +\,2ax & -\,3bc,\ \textit{Ans.}
\end{array}$$

2. Add $3a + 7x$; $2a - 5x$; $3x - 2a^2$; $ax + 5a$; $3x + a$.
3. Add $2a^{\frac{1}{3}} + 3a^{\frac{1}{3}}$; $2ab^{\frac{1}{3}} - a^{\frac{1}{3}}$; $a^{\frac{1}{3}} - ab$.
4. Add $5a^2b + 7ab^2$; $-7ax^2 + a^2b$; $-ax^2 + 3ab^2 - a^2x$.
5. Add $3ab^2 - 4a^2b + a^3$; $5ab^2 - 4ac^2 - c^3$; $2a^2b - 7ab^2 - 6ac$.
6. Add $4xy^2 + 4x^2y$; $5x^2y + 2xy^2$; $7x^2y^2 - 3xy^2$.
7. Simplify $ax - bx + 2ax - bx + cx - 2ax$.
8. Simplify $a^{\frac{1}{2}}b^{\frac{1}{2}} + ab^{\frac{1}{2}} - a^{\frac{1}{2}}b + 2a^{\frac{1}{2}}b^{\frac{1}{2}} - 2ab^{\frac{1}{2}}$.
9. Simplify $axy^2 + 2ax^2y - 3ay^2x - x^2ay$.

10. Simplify $3\sqrt{ab} + 2\sqrt{ac} - 2(ab)^{\frac{1}{2}} - a^{\frac{1}{2}}c^{\frac{1}{2}}$.
11. Simplify $2ab + 3a^2b - 3ab + 2b$.
12. Simplify $3xy^2 + 2ay^2 - 3by^2$.
13. Simplify $mn - 2an + bn$.

110. ***Similar Polynomial Terms*** **may be united as well as monomials; as,**

14. Add $2(x+y) + 3(x-y)$; $3(x+y) - (x-y)$.
Ans. $5(x+y) + 2(x-y)$.
15. Add $2(a+b)^{\frac{1}{2}} + 3\sqrt{a-b} - \sqrt{a+b} - 4(a-b)^{\frac{1}{2}}$.
16. Add $a\sqrt{x-1}$; $-b\sqrt{x-1}$; $-c(x-1)^{\frac{1}{2}}$.
17. Add $2(a-x)^{\frac{1}{2}} - 4\sqrt{a-x} + 3(a-x)^{\frac{1}{2}}$.

SUBTRACTION.

111. ***Subtraction*** is taking one quantity from another.

The ***Minuend*** is the quantity from which the subtraction is made.

The ***Subtrahend*** is the quantity subtracted.

The ***Difference*** is the result of subtraction.

112. Algebraic subtraction depends upon the following

PRINCIPLES.

1°. *Similar quantities only can be subtracted one from another.*

2°. *Subtracting a positive quantity is equivalent to adding an equal negative one.*

3°. *Subtracting a negative quantity is the same as adding an equal positive one.*

4°. *The sum of the difference and subtrahend is equal to the minuend.*

113. The only difference between Subtraction and Addition in Algebra is that when we propose to *subtract* a quantity, we give that quantity another *negative* sign, and therefore change all its signs. Hence the following

GENERAL RULE.

114. *Change the signs of quantities to be subtracted, and unite the terms as in addition.* (Art. 107.)

NOTES.—1. For convenience, similar terms may be written under each other, as in addition. (Art. 109.)

2. When a result is to be found from several quantities by adding some and subtracting others, the operation may be made one by the above rule.

3. In finding the difference between two quantities, it is *immaterial* which is made the subtrahend. The result will be the same in either case, except its sign.

Thus, the difference between 7 and 4 is $7 - 4 = 3$, or $4 - 7 = -3$.

When, therefore, a problem requires only the difference, without regard to sign, either quantity may be made the subtrahend.

EXAMPLES.

1. From $a^2 + 2ax + x^2$ take $a^2 - 2ax + x^2$.
2. From $3x^2 - 2x + 5$ take $x^2 - 2 + x$.
3. From $3ab + b^2$ take $-2ab + b^2$.
4. From $a^2 + 2ax + x^2$ take $ax - a^2 + x^2$.
5. From $a(x + y) - b(x + y)$ take $(x + y) - b(x - y)$.
6. From $3\sqrt{x^2 + a^2}$ take $5\sqrt{x^2 + a^2}$.
7. From $a + b - c + d$ take $a - b + c + d$.

Simplify the following:

8. $3x^3 - [(a + bx^2 - x^3) - (2x^3 + 2bx^2 + a)]$.
9. $3(ax^2 - ab^2) - 3a(x^2 - b^2)$.
10. $(mx)^{\frac{1}{2}} + m^{\frac{1}{2}}x^{\frac{1}{2}} - 2\sqrt{mx}$.

11. $2ab - 2b^2 + 2b(a + b)$.

12. $a + b - c - m + n - (a - b + c + m + n)$.

13. $2ab - ba - (ab + b^2)$.

USE OF PARENTHESES.

115. It will be observed in the above examples that the use of ***Parentheses*** is

1st. To bring several terms under the influence of the sign $-$.

2d. To connect several terms to the same coefficient.

3d. To subject several factors to the same exponent.

116. These parentheses may be removed in the several cases:

1st. By applying the sign $-$ to each term separately, or, what is the same thing, by changing the sign of each term.

2d. By connecting the coefficient with each term.

3d. By applying the exponent to each factor.

117. The parenthesis or vinculum, when used to save the repetition of some sign, coefficient, or exponent, can generally be removed without changing the value of the expression:

By applying the sign, coefficient, or exponent to each of the quantities separately to which it belongs.

118. In doing this, it must be remembered that coefficients belong to *terms*, and exponents to *factors*. We must not therefore apply an exponent to the several terms nor a coefficient to the several factors of the quantity within the parenthesis.

Thus, the expression $2(ab + b^2)$ does not equal $2a \cdot 2b + 2b^2$ (applying the 2 to both factors a and b), but $2ab + 2b^2$.

So $(a - b)^2$ is not the same as $a^2 - b^2$.

NOTE.—The parenthesis is here used to indicate that $a - b$ is used as a single factor, and it cannot be removed in the manner explained above.

119. Remove the signs of abbreviation from the following, and reduce to the simplest form:

1. $2\{a - [b + (c + 2x) - (y - z)]\}$.
2. $3[a - (b + c) + 2(bx - 1) - (bx + 1)]$.
3. $(a + b - c) - (b - a + c)$.
4. $\begin{array}{r|r|r|l} a & x - a & x + 2a & x. \\ + b & - c & + \ b & \\ - c & & & \end{array}$
5. $3\overline{abc}^{\frac{1}{2}} - 2\sqrt{abc} + (2abc)^{\frac{1}{2}}$.
6. $a[a - a(b - c)] - (a^2 + a^2b) + a^2c$.
7. $\sqrt{ab} + a^{\frac{2}{3}}b^{\frac{2}{3}} - 3xy^2 -^{2}\left(a^{\frac{2}{3}}\sqrt[3]{b^2} + 2xy^2 - a^{\frac{1}{2}}b^{\frac{1}{2}}\right)$.
8. $x -^{3}\left\{\left[a^{\frac{3}{5}}b^{\frac{3}{5}} + 2ab - (ab)^{\frac{1}{2}}\right]\right.$
   $\left. - \left[ab + a^{\frac{1}{2}}b^{\frac{1}{2}} - \left(\sqrt[5]{(ab)^3} - x\right)\right]\right\}$.
9. $(a + b - c)\sqrt{x + y} - (a + b + c)(x + y)^{\frac{1}{2}}$.
10. $(a + 2a^{\frac{1}{2}}b^{\frac{1}{2}} + b) - (a - 2\sqrt{ab} + b)$.
11. $(a - 2a^{\frac{1}{2}}b^{\frac{1}{2}} + b) -^{3}(a + 2\sqrt{a}\sqrt{b} \pm b)$.
12. $m(a + b) - m(a - b) + 2m(b - a)$.

120. In the present chapter we have considered quantities as *Terms*. It now remains to treat of them as *Factors*, in connection with the following subjects, viz.:

Multiplication, Division, Factoring, Greatest Common Divisors, Least Common Multiples, Fractions, Powers, and Roots.

CHAPTER III.

MULTIPLICATION.

121. ***Multiplication*** is the process by which the *sum* of *any number of equal terms* is found *without performing the addition.*

The ***Multiplicand*** is the *quantity multiplied.*

The ***Multiplier*** is the *number* by which it is *multiplied.*

The ***Product*** is the *result* of multiplication.

The *multiplier* and *multiplicand* are called ***Factors.***

122. To illustrate the difference between *multiplication* and *addition,* take the following example:

What is the sum of four distances, each equal to 5 feet?

We may find the answer to this by saying mentally, 5 and 5 are 10, and 5 are 15, and 5 are 20; or we may say, 4 times 5 are 20. The mental work of the first is that of successive additions; of the second it consists simply in *giving the required sum from memory,* knowing the *term* and *number of times* it is used.

In addition, if one does not remember what *number* 5 and 5 make, he may begin at 5 and *count,* adding *one* at a time, and thus find the *sum.*

In *multiplication,* the result can be given *only by memory,* which associates *certain products* with *certain factors,* and if the memory fail, the answer can be found *only by addition.*

In the example above, when the 20 is considered as made up by the *addition* of four fives, it is called *a sum;* but when it is found by considering the numbers 5 and 4, and giving *from memory* the result without adding, it is called *a product.*

The two operations are expressed thus:

1st.	$5 + 5 + 5 + 5 = 20.$
2d.	$5 \times 4 = 20.$

In the first, the 5's are *terms* and the 20 is their *sum.* In the second, 5 and 4 are *factors* and 20 is their product. (Art. 11.)

NOTE.—It is important that the distinction between terms and factors should be kept in mind:

1st. *Terms are quantities united by **algebraic addition.***

2d. ***Factors** are quantities **one of which shows how many times the other is used as a term.***

123. A product formed by *two factors* may be *multiplied* by a *third* factor, and this product by a *fourth,* and so on indefinitely. A product may therefore contain any number of factors. Hence the definition:

124. ***Multiplication** is the process of combining **factors into a product.***

PRINCIPLES.

125. 1°. *The multiplier must be considered an abstract quantity.*

2°. *The product is of the same nature as the multiplicand; for, repeating a quantity does not alter its nature.*

3°. *The product of two or more factors is the same in whatever order they are multiplied.*

126. Multiplication may be considered under two cases:

1st. Multiplication of monomials.

2d. Multiplication of polynomials.

CASE I.

127. Multiplication of Monomials.

By algebraic notation (Art. 53), we have for the multiplication of monomials the following

RULE.—*To the product of the numerical coefficients annex the literal factors, giving each an exponent equal to the sum of all its exponents in the several factors.*

NOTE.—This rule applies to any number of monomial factors.

128. The ***Signs*** of the ***Factors*** in multiplication are to be treated as *factors of direction,* and must enter the product precisely as the other factors. (Art. 91.) Hence,

129. For the ***Signs*** in Multiplication we have the following

RULE.—*An* ***even*** *power of* — ***is positive;*** *an* ***odd*** *power* ***negative.***

Or, *An even number of negative factors gives a positive product, an odd number a negative.*

130. When there are only two factors, this gives the common

RULE.—*Like signs give* + ; *unlike,* —.

EXAMPLES.

Find the product of the following:

1. $2ab^2x \times 3a^2b^3x^2$. *Ans.* $6a^3b^5x^3$.
2. $3a^2xy \times 2axy^2 \times 5a^3x^2 \times \frac{1}{2}abc$. *Ans.* $15a^7bcx^4y^3$.
3. $\frac{2}{3}a^{\frac{1}{2}}x^{\frac{1}{3}}y^2 \times \frac{5}{4}x^{\frac{2}{3}}yz \times a^{\frac{5}{2}}x$.
4. $\frac{3}{4}a^{\frac{3}{2}}b^{\frac{2}{3}}c \times \frac{4}{3}a^{\frac{3}{2}}b^{\frac{2}{3}}x \times \frac{1}{2}a^{\frac{5}{6}}b^{\frac{1}{3}}x^2$.
5. $7a^2x^{\frac{1}{2}}y^3 \times 3ax^2y \times abx$.
6. $5a^mb^nx \times 2a^rb^sz \times \frac{1}{2}ab$.
7. $2a^{\frac{m}{n}}b^{\frac{n}{m}} \times 3a^mb^n \times a^2b^2$.
8. $15a^{1-m}b^{1-n} \times \frac{1}{5}a^mb^n \times ax$.
9. $12a^{1-m}b^{1-n} \times \frac{1}{3}a^{m-1}b^{n-1}$.
10. $a^{m+n-1}b^{n-m+1} \times a^{m-n+1}b^{n+m-1}$.
11. $a^{m+n}b^{m-n} \times a^{m-n}b^{m+n}$.
12. $(ab)^{m+n} \times a^{m-n}b^{m-n}$.
13. $x^{m-1}y^{n-2} \times xy^2$.
14. $x^{n-1}y^n \times x^ny^{1-n}$.
15. $x^{n^2}y^{m^2} \times x^{n(n-2)}y^{m(m-1)}$.
16. $2a^2x \times -3x^{\frac{1}{3}} \times -ax^2 \times ab$.
17. $ab^2 \times \pm a^2x \times -bx^2 \times abx$.
18. $\sqrt{ab} \times -a^{\frac{1}{2}}b^{\frac{1}{2}} \times (-ax)^2 \times 2ab$.
19. $3\sqrt[3]{ax} \times -2\sqrt{ax} \times (-a^{\frac{1}{2}}x^{\frac{1}{2}}) \times ax^2$.
20. $(-ax^3)(-a^3x)(a^2x^2)\sqrt{ax}$.

CASE II.

131. Multiplication of Polynomials.

The ***Multiplication of Polynomials*** is performed by the following

RULE.—*Multiply each term of the multiplicand by each term of the multiplier, and add the products.*

NOTES.—1. This does not differ in principle from the method of multiplying numbers, where each figure is multiplied separately and the products added. The multiplier may be a monomial,

2. For *convenience* in adding the partial products, *like* terms should be placed under each other.

3. The *multiplication* of polynomials may be *indicated* by inclosing each factor in a parenthesis, and writing one after the other. Thus, $(a+b+c)(a+b+c)$ is equivalent to $(a+b+c)\times(a+b+c)$.

21. Multiply $x^2 - 2ax + a^2$ by $a + x$.

SOLUTION.—We write the multiplier under the multiplicand, and proceed thus:

OPERATION.

$$\begin{array}{lrrrrl}
 & & x^2 & -\ 2ax & +\ a^2 & \\
 & & & a & +\ x & \\
\hline
\text{Multiplying by } a, & & ax^2 & -\ 2a^2x & +\ a^3 & \\
\text{Multiplying by } x, & -\ 2ax^2 & +\ a^2x & & & +\ x^3 \\
\hline
\text{Adding products,} & -\ ax^2 & -\ a^2x & +\ a^3 & +\ x^3, & \textit{Ans.}
\end{array}$$

Multiply the following:

22. $(a^2 + 2ax + x^2)(a^2 - 2ax + x^2)$.

23. $2ab(a^2 - ax^2 - b^2)$.

24. $(a + x)(a^2 + x^2)(a^2 - x^2)$.

25. $(2ax - x^2)(a + bx + cx^2)$.

26. $(2a + bx - cx^2)(2a - bx + cx^2)$.

27. $(a^m + b^n)(a^m + b^n)(a^m - b^n)$.

28. $(1 + x)(1 - x)(1 + x - x^2)(1 - x + x^2)$.

MULTIPLICATION BY DETACHED COEFFICIENTS.

132. When the terms of the *multiplicand* and *multiplier* can both be arranged so that the exponents of each letter increase or decrease by unity in the successive terms, we may write simply the coefficients, and afterwards introduce the letters into the product.

29. Multiply $a^2 + 2ax - x^2$ by $a + x$, by detached coefficients.

SOLUTION.—We write

$$\begin{array}{r}
1 + 2 - 1 \\
1 + 1 \\
\hline
1 + 2 - 1 \\
1 + 2 - 1 \\
\hline
1 + 3 + 1 - 1
\end{array}$$

$a^3 + 3a^2x + ax^2 - x^3$, *Product.*

By comparing this with the multiplication of the same quantities by the ordinary method, the process will be made plain.

133. By this method of detached coefficients, a term must be introduced with 0 for its coefficient when necessary to make the exponents change by unity in the successive terms.

For example, to multiply $a^3 + 2ax^2 - x^3$ by $a^2 - x^2$, we must introduce a term between the first and second in each factor; thus,

$$\begin{array}{r}
1 + 0 + 2 - 1 \\
1 + 0 - 1 \\
\hline
1 + 0 + 2 - 1 \\
- 1 - 0 - 2 + 1 \\
\hline
1 + 0 + 1 - 1 - 2 + 1
\end{array}$$

Product, $a^5 + 0a^4x + a^3x^2 - a^2x^3 - 2ax^4 + x^5$,

from which the second term, having 0 for a coefficient, may be omitted.

EXAMPLES.

Find the product of the following:

30. $(3a^5 - 2a^4b + 7a^2b^3 - 5b^5)(a^2b - ab^2)$.
31. $(a^3 - 2ax^2 + x^3)(a^2 - x^2)$.
32. $(x^4 - y^4)(x^2 + y^2)$.
33. $(a^6 - 5a^2x^4 + 3ax^5 - x^6)(a^2 - ax)$.

CHAPTER IV.

DIVISION.

134. ***Division*** is finding *how many times one quantity is contained in another;* or, finding the *measure* of a quantity with a given quantity as a *unit of measure.*

The ***Dividend*** is the *quantity to be divided, or measured.*

The ***Divisor*** is the *quantity by which we divide,* or the *unit of measure.*

The ***Quotient*** is the *number* found by division.

135. ***Division*** is the *reverse* of ***Multiplication,*** the *dividend* being the *product,* the *divisor* and *quotient* the *factors.*

Multiplication combines factors into a *product; Division removes factors* from a *product.*

Multiplication finds the *sum* of a *given number* of *equal terms; Division* finds *how many times* a *given term must be taken,* or *what term* must be taken a *given number of times,* to produce a given *sum.*

136. The relations of *Dividend, Divisor,* and *Quotient* give us readily the following

PRINCIPLES.

1°. *Multiplying or dividing the dividend multiplies or divides the quotient.*

2°. *Multiplying or dividing the divisor divides or multiplies the quotient.*

3°. *Multiplying or dividing both dividend and divisor by the same factor does not affect the quotient.*

137. We have *three cases* in *Division:*

1st. Dividing a *monomial* by a *monomial.*

2d. Dividing a *polynomial* by a *monomial.*

3d. Dividing a *polynomial* by a *polynomial.*

CASE I.

138. To Divide a Monomial by a Monomial.

The ***Division of Monomials*** is performed by the following

RULE.—*Divide the numerical coefficient of the dividend by that of the divisor, and annex to the quotient the letters of both dividend and divisor, giving each an exponent found by subtracting its exponent in the divisor from its exponent in the dividend.*

NOTE.—The *reason* of this rule is plain. Since the factors of the divisor are to be taken from the dividend, it is evident that any factor will be found in the quotient as many times as it is found in the dividend minus the number of times it is found in the divisor.

EXAMPLES.

1. Divide $8a^2bc^3x^4$ by $2acx^2$.

SOLUTION.—Dividing the coefficient 8 by 2, we have 4, to which we annex the letters; thus, $4abcx$. To find the *exponents* of these letters, we have for a, $2-1=1$; for b, $1-0=1$; for c, $3-1=2$; for x, $4-2=2$. Applying these exponents, and remembering that the exponent 1 need not be written, we have as the quotient $4abc^2x^2$.

2. Divide $7a^3b^2c^2x^3$ by a^2bc^2x.

SOLUTION.—Following the same order, we find $7abc^0x^2$. Here we have 0 for the exponent of c, showing that the factor c is not found in the quotient. We may therefore omit it and write, $7abx^2$.

3. Divide $5a^2$ by $5a^2$.

SOLUTION.—Dividing as before, we get 1 for the coefficient and zero for the exponent of a in the quotient. The quotient is therefore 1, as it should be, since the dividend and divisor are the same. If we choose, we may write a^0 and omit the coefficient 1.

NOTE.—It follows that any quantity with 0 for an exponent is equal to 1.

4. Divide $4a^2bc^3$ by $2ab^2c$.

SOLUTION.—Following the rule as before, we find that the exponent of b in the divisor is greater than in the dividend, and we have $1 - 2$ for its exponent in the quotient. This gives us -1, and we have the quotient $2ab^{-1}c^2$.

139. Here we find the sign of an exponent becomes *negative*, and we have to consider what meaning we are to attach to the *signs* $+$ and $-$ when applied to *exponents*.

In this example the exponent became negative because we had the factor b more times in the divisor than in the dividend, and we could not take away from the dividend more factors of that kind than it contained. We might have taken away such factors as were in the dividend, and expressed the division we could not perform. This would have given us $2ac^2 \div b$, or $\frac{2ac^2}{b}$.

We must therefore interpret the expression $2ab^{-1}c^2$ as meaning the same; that is, the *negative exponent* must be understood to express division (Art. 54), and ab^{-2} will equal $\frac{a}{b^2}$ or $a \div b^2$, and

$$b^{-2} = 1 \times b^{-2} = \frac{1}{b^2} = 1 \div b^2. \text{ Hence,}$$

140. We have this interpretation of the force of the signs $+$ and $-$ affecting an exponent:

A positive exponent indicates the use of factors as multipliers; a negative exponent, as divisors.

NOTE.—This is consistent with the *general definition* of these signs, by which they are made to show *opposite directions*, *multiplication* and *division* being *opposite* in direction, one *putting in* and the other *taking out* a factor. (Arts. 81–91.)

141. By the *analogy* between the coefficient and exponent, we see:

1st. The *coefficient* shows what *equal terms*, and the *exponent* what *equal factors*, are used.

2d. The *sign* of the *coefficient* shows whether the *equal terms*, and the *sign* of the *exponent* whether the equal factors are introduced or removed, the former by *addition* or *subtraction*, the latter by *multiplication* or *division*.

142. It must be understood, also, that if the sign $-$ be repeated before an exponent, it again *reverses* the direction in which the factors are used.

143. We may have what is equivalent to *two* negative signs, without the signs themselves.

Thus, when we say subtract $-a$, the word *subtract* is equivalent to the sign $-$, and the term a is to be added or is the same as $-^2 a = +a$. So also in $\frac{a}{b^{-2}}$, the b^2 is a multiplier, its position below the line being equivalent to the sign $-$ before its exponent, which with the sign already there gives $+$. It might therefore be written ab^2.

144. From the above we have

$$a^{-1} = 1 \times a^{-1} = \frac{1}{a}. \quad \text{That is,}$$

1°. *A quantity with a negative exponent is equal to the reciprocal of the same quantity with a like positive exponent.*

Again, $$a = 1 \times a = \frac{1}{a^{-1}}. \quad \text{That is,}$$

2°. *A quantity with a positive exponent is equal to the reciprocal of the same quantity with a like negative exponent.*

Again, $a \times \frac{1}{b} = a \div b$, and $a \div \frac{1}{b} = a \times b$. That is,

3°. *Multiplying or dividing by any quantity is the same as dividing or multiplying by its reciprocal.*

Divide the following:

5. $\dfrac{8a^2b^{-2}cd^3}{2abc^2d}$.
6. $4a^3bc^2d^{-1} \div 6a^4b^2cd$.
7. $\dfrac{9a^5b^{-2}c^2d}{3a^3bc^{-3}d^{-1}}$.
8. $15a^3b^2x^{-3}y \div 5a^{-2}b^4x^3y^{-1}$.

145. When the *signs* of the *dividend* and *divisor* are considered, they must be treated as the *other factors*, in accordance with the principles already established. (Art. 128.) That is,

The quotient will have the sign —, with an exponent equal to its exponent in the dividend minus its exponent in the divisor.

Take the following example:

9. Given $\dfrac{-4(-a^3)(-b^5)(-c^2)(-d^7)}{-2(a)(-b^2)(-c^2)(d^3)}$, to find the quotient.

SOLUTION.—Cancelling or removing the factors of this divisor from the dividend, we have $2a^2b^3d^4$.

In the same way, we may cancel *factors of direction*, remembering that the sign +, like the factor 1, has no power.

We have here five minus signs in the dividend and three in the divisor, or, what is the same thing, $-^5$ and $-^3$, leaving $-^2$, or + for the quotient. It will be readily seen that the same result would be given by adding the exponents of —, for whenever the difference of the exponents is odd, the sum will be odd, and when the difference is even, the sum will be even, hence we may use the same rule for the sign of the quotient in division as for the product in multiplication. (Art. 129.) Hence, the

RULE.—I. *An even number of negative signs gives a positive quotient, an odd number a negative.*

II. If there be but two signs:

Like signs give + ; unlike signs, —.

NOTE.—If it be asked why these signs are thus treated in multiplication and division, the answer is, that such use of them meets the wants of mathematical analysis, as will be abundantly illustrated in the solution of problems.

Divide the following:

10. $-8a^mb^n \div 2ab$.

11. $6a^5b^4 \div -3ab^2$.

12. $-5a^{m+n}b^{m-n} \div -a^nb^m$.

13. $-8a^{-1}b^{-2} \div 2a^{-2}b^{-4}$.

14. $2a^{-2}b^{-4} \div -8a^{-1}b^{-2}$.

15. $3a^{\frac{1}{2}}b^{\frac{1}{2}} \div -a^{\frac{1}{4}}b^{\frac{1}{4}}$.

16. $-a^{\frac{1}{2}}(-ab) \div b^{\frac{1}{2}}(a^{\frac{1}{2}}b^{\frac{1}{2}})$.

17. $-a^{\frac{m}{n}}b^{\frac{r}{s}} \div a^{\frac{m-n}{n}}b^{\frac{r+s}{s}}$.

18. $a^{m(m+n)}b^{n(m+n)} \div -(ab)^{mn}$.

19. $-a^{m^2}b^{n^2} \div -a^{m}(-b^{n})$.

20. $x^{2n}y^{2m} \div x^{n-1}y^{m+1}$.

21. $(abx)^{2n+1} \div a^{n-1}b^{n}x^{n+1}$.

CASE II.

146. To Divide a Polynomial by a Monomial.

A *polynomial* is divided by a *monomial* by the following

RULE.—*Divide each term of the polynomial by the divisor.*

NOTE.—Since *division* is the reverse of multiplication, the *reason* for this rule will appear from Art. 131.

EXAMPLES.

Find the value of the following:

1. $(2a^2b^3x - 4a^3b^2x^2 + 6a^4b^3x^3) \div 2a^2b^2$.
2. $(8a^nb^m + 4a^{2n}b^{3m} - 12a^{4n}b^{3m}) \div 4a^nb^m$.
3. $(9a^mb^m - 6a^nb^m + 12a^{2m}b^{2n}) \div 3a^nb^m$.
4. $(4a^{m-n}b^{s-n} - 6a^{m+n}b^{s+n}) \div 2a^{-n}b^n$.
5. Divide the same by $2a^nb^{-n}$.
6. Divide the same also by $2a^{n-m}b^{n-s}$, and by $2a^mb^s$.
7. Divide $x^y - y^x$ by xy.
8. Divide $a^{\frac{m+n}{m}} + b^{\frac{m+n}{n}}$ by ab.
9. Divide $a^{\frac{mn}{r}} + b^{\frac{mn}{s}}$ by $a^{\frac{m}{r}}b^{\frac{m}{s}}$.

CASE III.

147. To Divide a Polynomial by a Polynomial.

1. Divide $3ab^2 + 3a^2b + b^3 + a^3$ by $a + b$.

ANALYSIS.—1st. Since the dividend is the product of the divisor and quotient, that term of the dividend which has the highest power of a must have been produced by the multiplication of those terms in the divisor and quotient which contain the highest powers of a. If, therefore, we divide that term of the dividend containing the highest power of a by the corresponding term of the divisor, we shall find one term of the quotient.

2d. Multiplying the divisor by this term of the quotient and subtracting the product from the dividend, we shall have a remainder to be divided as before.

3d. Whenever this remainder becomes *zero*, the division will be complete.

4th. The division may be stopped at any time and the quotient be completed by adding the remainder over the divisor in the form of a fraction to indicate the uncompleted division.

It will be more convenient in dividing to arrange the dividend and divisor in the order of the ascending or descending powers of the same letter.

OPERATION.

Dividend,	$a^3 + 3a^2b + 3ab^2 + b^3$	$a + b$	Divisor.
1st product,	$a^3 + a^2b$	$a^2 + 2ab + b^2$	Quotient.
	$2a^2b + 3ab^2 + b^3$	. . . . .	1st remainder.
2d product,	$2a^2b + 2ab^2$		
	$+ ab^2 + b^3$	. . . . .	2d remainder.
3d product,	$+ ab^2 + b^3$		

148. We have from the above illustration the following

RULE.—I. *Arrange the terms of both dividend and divisor in accordance with the ascending or descending powers of the same letter.*

II. *Divide the first term of the dividend by the first term of the divisor, and write the result for the first term of the quotient.*

III. *Multiply the divisor by this term of the quotient, and subtract the product from the dividend.*

IV. *Divide this remainder as before, and so on till the division is complete or a remainder is found which has no term divisible by the first term of the divisor.*

V. *Write the final remainder, if any, over the divisor in the form of a fraction and add it to the quotient.*

EXAMPLES.

2. Divide $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$ by $a^2 - 2ab + b^2$.

OPERATION.

$$\begin{array}{ll|l}
 & a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5 & \underline{a^2-2ab+b^2} \quad \text{Divisor.} \\
 & \underline{a^5-2a^4b+\quad a^3b^2\qquad\qquad\qquad\qquad} & a^3-3a^2b+3ab^2-b^3, \quad \text{Quot.} \\
\text{1st rem.,} & \quad -3a^4b+\ 9a^3b^2-10a^2b^3 & \\
 & \quad \underline{-3a^4b+\ 6a^3b^2-\ 3a^2b^3\qquad} & \\
\text{2d rem.,} & \qquad\qquad 3a^3b^2-\ 7a^2b^3+5ab^4 & \\
 & \qquad\qquad \underline{3a^3b^2-\ 6a^2b^3+3ab^4\qquad} & \\
\text{3d rem.,} & \qquad\qquad\qquad -\ \ a^2b^3+2ab^4-b^5 & \\
 & \qquad\qquad\qquad \underline{-\ \ a^2b^3+2ab^4-b^5} &
\end{array}$$

3. Divide $x^6 - 2x^5 + 20x^3 - 33x^2 - 5x^4 + 14x - 3$ by $3 - 2x + x^2$.

4. Divide $x^8 - y^8$ by $x^2 - y^2$.

5. Divide $x^6 + a^6$ by $x^2 + a^2$.

6. Divide $a - x$ by $a^{\frac{1}{4}} - x^{\frac{1}{4}}$.

7. Divide $a^{\frac{5}{2}} - a^{\frac{3}{2}}x + ax^{\frac{3}{2}} - 2a^{\frac{1}{2}}x^2 + x^{\frac{5}{2}}$ by $a^{\frac{3}{2}} - ax^{\frac{1}{2}} + a^{\frac{1}{2}}x - x^{\frac{3}{2}}$.

8. Divide $x^{2n} - a^{2n}$ by $x^n - a^n$.

9. Divide $a^{m+n}b^n - 4a^{m+n-1}b^{2n} - 3a^{m+n-2}b^{3n} + 6a^{m+n-3}b^{4n}$ by $a^m - 3a^{m-1}b^n - 6a^{m-2}b^{2n}$.

10. Divide $a^{-1} + b^{-1}$ by $a^{-\frac{1}{3}} + b^{-\frac{1}{3}}$.

11. Divide $a^5 - 5a^4x + 2a^3x^2 - 5a^2x^3 + ax^4$ by $a^2 + x^2$.

12. Divide $x^2y - 3xy - x + 2xy^2$ by $xy - 3y - 1 + 2y^2$.

13. Divide $1 - 3x + 3x^2 - x^3$ by $1 - x$.
14. Divide $b^3 + 3bx^2 + 3b^2x + x^3$ by $b + x$.
15. Divide $x^4 + 5y^4 - 6x^2y^2$ by $x - 3y$.
16. Divide $6a^4 + 4a - 10a^3 - 15$ by $3a^2 - 2a + 1$.
17. Divide $2a^4 + 11a^3x + 20a^2x^2 + 13ax^3 + 2x^4$
by $a^2 + 3ax + 2x^2$.

18. Divide $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$ by $a^2 - 2ab + b^2$.

ABBREVIATED OPERATION.

$$
\begin{array}{r|rrrrr}
a^2 & a^4 & -\,4a^3b & +\,6a^2b^2 & -\,4ab^3 & +\,b^4 \\
+\,2ab & & +\,2a^3b & -\,4a^2b^2 & +\,2ab^3 & \\
-\,b^2 & & & -\,a^2b^2 & +\,2ab^3 & -\,b^4 \\
\hline
 & & -\,2a^3b & +\,a^2b^2 & +\,0 & +\,0 \\
\hline
\text{Quotient,} & a^2 & -\,2ab & +\,b^2 & &
\end{array}
$$

ANALYSIS.—1st. Divide a^4 by a^2 and set down the quotient a^2.

2d. Multiply the several terms of the divisor, except the first, by this quotient, and set the products $2a^3b$ and $-a^2b^2$ as above.

3d. Add the second column and set the sum $-2a^3b$ below.

4th. Divide this by the first term of the divisor, and proceed as before.

149. By this method, the *dividend* and *divisor* must be arranged in accordance with the *ascending* or *descending* powers of *each letter*; that is, the *exponent of each letter* must either increase or decrease by unity from left to right in both *dividend* and *divisor*.

Terms may be inserted with 0 for coefficients, if by so doing the exponents may be made to form an increasing or decreasing series.

It will also be observed that the *signs* of the divisor, except the first, have been *changed*, which enables us to add the products instead of subtracting.

The product obtained by multiplying the first term of the divisor is not written, since it always *cancels* a term of the dividend.

The other products are written each under a similar term of the dividend, and in line with that term of the divisor from which it was obtained.

Let the student work in this way such examples as are suited to this process, found in Art. 148.

DIVISION BY DETACHED COEFFICIENTS.

150. The work of division may often be abbreviated by dropping the literal factors and replacing them in the quotient. This can only be done when the dividend and divisor can be arranged according to the *ascending* or *descending* powers of each letter, as in multiplication. (Art. 132.)

1. Divide $a^4 - b^4$ by $a - b$.

This may be written

$$a^4 + 0a^3b + 0a^2b^2 + 0ab^3 - b^4,$$

in which the exponents of a decrease and those of b increase by unity in the successive terms. Writing the coefficients only, and dividing, we have,

$$\begin{array}{l|l}
1 + 0 + 0 + 0 - 1 & 1 - 1 \quad \text{Coef. of Divisor.} \\
\underline{1 - 1} & \overline{1 + 1 + 1 + 1} \quad \text{Coef. of Quot.} \\
\quad + 1 + 0 & \\
\quad \underline{\;\;\, 1 - 1} & \\
\qquad\quad 1 + 0 & \\
\qquad\quad \underline{1 - 1} & \\
\qquad\qquad\quad 1 - 1 & \\
\qquad\qquad\quad \underline{1 - 1} &
\end{array}$$

It is evident by dividing a^4 by a that the first term of the quotient will be a^3, and we may then write

$$a^3 + a^2b + ab^2 + b^3, \quad \textit{Ans.}$$

EXAMPLES.

Divide by detached coefficients:

2. $2a^3 - 6a^2x + 6ax^2 - 2x^3$ by $2a - 2x$.
3. $a^3 - b^3$ by $a^2 + ab + b^2$.
4. $a^4 - x^4 + 2ax^3 - a^2x^2$ by $ax + a^2 - x^2$.
5. $x^4 - 4x^3 + 6x^2 - 4x + 1$ by $x - 1$.
6. $x^5 + x^4y - 5x^3y^2 + 6xy^4 + 2y^5$ by $x^2 + 3xy + y^2$.
7. $1 - 4b + 10b^2 - 16b^3 + 17b^4 - 12b^5$ by $1 - 2b + 3b^2$.
8. $2a^4 + 11a^3x + 20a^2x^2 + 13ax^3 + 2x^4$ by $a^2 + 3ax + 2x^2$.

SYNTHETIC DIVISION.

151. The operation may be still further shortened, when the first coefficient of the *divisor* is unity, by using detached coefficients, in the manner of Art. 148, Ex. 18.

5. Divide $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$ by $a^2 - 2ab + b^2$.

OPERATION.

$$\begin{array}{r|l}
1 & 1 - 4 + 6 - 4 + 1 \\
+2 & \quad\; + 2 - 4 + 2 \\
-1 & \qquad\quad - 1 + 2 - 1 \\
\hline
 & 1 - 2 + 1
\end{array}$$

$\therefore$ Quotient $= a^2 - 2ab + b^2$.

NOTE.—Observe that the coefficient of each remainder becomes the coefficient of a term of the quotient. This method is called *Synthetic Division.**

In like manner divide the following:

6. $a^6 + 2a^3b^3 + b^6$ by $a^2 - ab + b^2$.
7. $x^3 + x^2y - xy^2 - y^3$ by $x - y$.
8. $a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5$ by $a^2 + 2ax + x^2$.

The method of *Synthetic Division* is especially convenient in case of a binomial divisor. Thus,

9. Divide $x^5 - 2x^4 + 5x^3 + 3x^2 - 12x - 28$ by $x - 2$.

OPERATION.

$$\begin{array}{r|l}
1 & 1 - 2 + 5 + 3 - 12 - 28 \\
+2 & \quad\; + 2 + 0 + 10 + 26 + 28 \\
\hline
 & 1 + 0 + 5 + 13 + 14
\end{array}$$

$\therefore\ x^4 + 0x^3 + 5x^2 + 13x + 14 = x^4 + 5x^2 + 13x + 14 =$ Quotient.

Or, transferring the divisor to the right of dividend, and omitting the first term, an arrangement frequently more convenient, we shall have

$$\begin{array}{l|l}
1 - 2 + 5 + 3 - 12 - 28 & \underline{2} \\
\quad\; + 2 + 0 + 10 + 26 + 28 & \\
\hline
1 + 0 + 5 + 13 + 14 &
\end{array}$$

And $x^4 + 5x^2 + 13x + 14 =$ Quotient.

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* It was first proposed by W. G. Horner, of England. (See p. 307, Note 6.)

10. Divide $x^5 - 32$ by $x - 2$.

$$\begin{array}{l|l} 1 + 0 + 0 + 0 + \ 0 - 32 & \underline{2} \\ \underline{\quad + 2 + 4 + 8 + 16 + 32} & \\ 1 + 2 + 4 + 8 + 16 & \end{array}$$

$x^4 + 2x^3 + 4x^2 + 8x + 16$, *Ans.*

11. Divide $x^4 - 2x^2 + 3x - 7$ by $x + 3$.

$$\begin{array}{l|l} 1 + 0 - 2 + \ 3 - \ 7 & \underline{-3} \\ \underline{\quad - 3 + 9 - 21 + 54} & \\ 1 - 3 + 7 - 18 + 47 & \end{array}$$

$x^3 - 3x^2 + 7x - 18 + \dfrac{47}{x+3}$, *Ans.*

NOTE.—In this case we have the remainder 47, and therefore the division is incomplete.

Divide in like manner:

12. $x^6 - 3x^4 + 2x^2$ by $x - 1$.
13. $x^5 + 7x^4 - 3x^2 + 7x - 5$ by $x - 3$.
14. $x^4 - 2x^3 + 8x - 16$ by $x - 2$; also by $x + 2$.
15. $x^5 - 5x^3 + 2x^2 - 7$ by $x + 1$.
16. $x^7 - 5x^5 + 2x^3 - 1$ by $x + 3$.
17. $x^6 - 4x^5 + 6x^4 - 6x^2 + 4x - 1$ by $x - 1$ and by $x + 1$.
18. $x^5 - 7x^3 + 6x - 5$ by $x + 2$ and by $x - 3$.
19. $x^8 - 1$ by $x - 1$ and by $x + 2$.
20. $x^7 - 1$ by $x - 1$ and by $x + 1$.
21. $a^{12} + x^{12}$ by $a^4 + x^4$ and by $a^3 + x^3$.
22. $x^{10} + y^{10}$ by $x^2 + y^2$ and by $x^5 + y^5$.
23. $x^{14} + 1$ by $x^2 + 1$ and by $x^7 + 1$.
24. $x^{18} + a^{36}$ by $x^6 + a^{12}$ and by $x^3 + a^4$.
25. $x^9 + y^{12}$ by $x^3 + y^4$; also, $a^5 + b^{10}$ by $a + b^2$.
26. $x^5 - 6x^4 + 5x^3 - 7x^2 - 4x + 9$ by $x - 3$ and $x + 3$.
27. $x^7 + 3x^5 - 2x^4 - 5x^3 + 3x + 1$ by $x + 1$ and $x - 1$.
28. $x^6 + 4x^5 - 7x^4 + 2x^3 - 7x^2 + 4x + 16$ by $x + 1$ and $x - 1$.

CHAPTER V.

FACTORING.

152. ***Factoring*** is the process of *separating a quantity into factors.*

Unity having no power as a factor, will not be considered as a factor in the following discussions.

A ***Prime Factor*** is one which does not contain other factors.

Factors are ***prime to each other*** when they have *no common factor.*

153. The operation of factoring is performed either *by inspection,* in which we employ our previous knowledge of the forms of products; or *by trial,* in which we determine *by division* whether one quantity is a factor of another, and at the same time, discover the other factor.

NOTE.—Finding the *equal factors* of a quantity is the work of *Evolution,* which will be treated in its place. We are at present concerned especially with *unequal factors,* though equal factors may often be found by the same methods.

154. The ***Factors of a Monomial*** are too easily discovered to need explanation, since they are all indicated by the exponents. The same may be said of the *monomial factors* of a *polynomial,* since they must be factors of *each term* of the polynomial.

Thus, a^2b is a factor of $2a^3b + 3a^2b^3 - 5a^3b^3$, and we may write $a^2b(2a + 3b^2 - 5ab^2)$.

155. The ***Factoring of Binomials*** will be facilitated by the following theorems:

THEOREM I.—*The product of the sum and difference of any two quantities is equal to the difference of their squares.*

DEMONSTRATION. $(a + b)(a - b) = a^2 - b^2$.

THEOREM II.—*The difference of any two quantities is a factor of the difference of any like positive integral powers of the same quantities.*

DEMONSTRATION.—Let a and b represent the quantities, and a^n and b^n the like powers, in which n is a positive integer. We are to prove that $a^n - b^n$ is divisible by $a - b$.

$$\begin{array}{llr} \text{Dividing,} & a^n - b^n & \underline{|\,a-b} \\ & \underline{a^n - a^{n-1}b} & a^{n-1} \\ \text{We have the remainder,} & a^{n-1}b - b^n & \\ & = (a^{n-1} - b^{n-1})\,b. & \end{array}$$

One of the factors of the remainder is the difference of like powers of the same quantities, whose exponent is one less than in the dividend. If this factor be divisible by $a - b$, the dividend is also divisible by it.

We have therefore proved that if the Theorem be true for one value of n, it is also true for a value one unit greater.

For example, if it be true when $n=2$, it will also be true when $n=3$, and for the same reason when $n = 4, 5, 6$, etc.

But we know that $a - b$ is a factor of $a^2 - b^2$; that is, the theorem is true when $n = 2$. Hence, it is universally true.

COR. 1.—*The sum of two quantities is a factor of the difference of any like even positive integral powers of the same.*

That is, $a^{2n} - b^{2n}$ is divisible by $a + b$. For,

$$a^{2n} - b^{2n} = (a^2)^n - (b^2)^n,$$

which by the Theorem is divisible by

$$a^2 - b^2 = (a + b)(a - b).$$

COR. 2.—*The difference of any two powers whose exponents have a common factor, may be separated into factors one of which is a binomial.*

For, $a^{mn} - b^{nr} = (a^m)^n - (b^r)^n$, which has $a^m - b^r$ as a factor.

THEOREM III.—*The sum of any two quantities is a factor of the sum of any like odd positive integral powers of the same.*

That is, $a^{2n+1} + b^{2n+1}$ is divisible by $a + b$. For, dividing,

$$\begin{array}{llr} & a^{2n+1} + b^{2n+1} & \underline{|\,a+b} \\ & \underline{a^{2n+1} + a^{2n}b} & a^{2n} \\ \text{We have the remainder,} & -a^{2n}b + b^{2n+1} = (a^{2n} - b^{2n})(-b). & \end{array}$$

But $a^{2n} - b^{2n}$ has been shown to be divisible by $a + b$; hence, $a^{2n+1} + b^{2n+1}$ is also divisible by $a+b$.

COR.—*The sum of any two powers each of whose exponents has the same odd factor, may be separated into factors one of which is a binomial.*

That is, $a^{(2n+1)m} + b^{(2n+1)r}$ has a binomial factor. For this may be written, $(a^m)^{2n+1} + (b^r)^{2n+1}$, which has the factor $a^m + b^r$.

For example, $a^6 + b^9 = (a^2)^3 + (b^3)^3 = (a^2 + b^3)(a^4 - a^2b^3 + a^6)$.

It is evident that m and r may be equal; as, $a^6 + b^6 = (a^2)^3 + (b^2)^3$, which has the factor $a^2 + b^2$.

They may also be negative; as, $a^{-5} - b^{-10} = (a^{-1})^5 - (b^{-2})^5$, which has the factor $a^{-1} - b^{-2}$.

Factor the following:

1. $a^4 - b^4$. *Ans.* $(a^2 + b^2)(a - b)(a + b)$.
2. $a^3 - b^3$. *Ans.* $(a - b)(a^2 + ab + b^2)$.
3. $a^3 + b^3$. *Ans.* $(a + b)(a^2 - ab + b^2)$.
4. $a^{-4} - b^{-4}$.
5. $a^{-3} - b^{-3}$.
6. $a^{-3} - b^3$.
7. $a^6 - b^{-6}$.
8. $a^6 + b^6$.
9. $a^{12} + b^{15}$.
10. $a^{12} - b^{12}$.
11. $x^6 + y^9$.
12. $x^{-6} - y^{-9}$.
13. $x^6 + y^{-9}$.
14. $x^{10} + y^{15}$.
15. $x^{12} + y^{21}$.
16. $x^{-12} - y^9$.
17. $x^{14} + y^{14}$.
18. $x^{14} - 1$.
19. $1 + a^{10}$.
20. $1 - a^{-10}$.
21. $a^{72} - b^{96}$.
22. $x^{100} + y^{100}$.

156. To find the *binomial* or *polynomial* factors of a polynomial, the forms of products must be observed. The following theorem will aid in this inspection:

THEOREM IV.—*The square of a polynomial is equal to the sum of the squares of its terms plus twice the sum of the products of the terms taken two and two.*

DEMONSTRATION. $(a + b + c)^2 = a^2 + 2ab + 2ac + b^2 + 2bc + c^2$, as is shown by performing the multiplication, and the process is such as to make it evident that the same would apply to a polynomial of any number of terms. Hence,

COR. 1.—*The square of the sum of two quantities equals the sum of their squares plus twice their product; and the square of the difference equals the sum of their squares minus twice their product.* That is,

$$(a + b)^2 = a^2 + 2ab + b^2, \quad \text{and}$$

$$(a - b)^2 = a^2 - 2ab + b^2.$$

COR. 2.—*If two polynomials having like terms, but with unlike signs be multiplied together, the product will be the same as the square of one of the polynomials;* except,

1st. The *squares* of the terms having *unlike signs* in the two polynomials will be *negative.*

2d. The *double products* formed by those terms which have *like signs* in one polynomial and *unlike* in the *other* will not be found in the result; and the double products formed by terms having *like signs* in *each polynomial,* but *unlike* in the *two,* will be *negative.*

157. From this theorem and its corollaries factor the following examples:

23. $4a^2 - 9b^2 + c^2 + 4ac.$

ANALYSIS.—The terms which are perfect squares suggest the terms of the factors sought. These give $2a$, $3b$, and c. The only double product found is $4ac$; hence, $2a$ and c have either like or unlike signs in both factors, but $2a$ and $3b$, and $3b$ and c have unlike signs in one and like signs in the other factor. This requires for the factors, $2a - 3b + c$ and $2a + 3b + c$.

24. $4a^2 - 9b^2 + 6bc - c^2.$

25. $4a^2 - 4ac - 9b^2 + c^2.$

26. $a^2 + 2ab + b^2 + ac + bc.$

Observe that the first three terms are the square of $a+b$, which is also a factor of the last two terms.

27. $4 + 4b + b^2 + 2c + bc.$

28. $a^2 + 6a + 8.$

Separate into terms thus: $a^2 + 4a + 4 + 2a + 4.$

29. $a^2x^2 + 2ax^3 + x^4 - a^4 + 2a^2x - x^2$.

30. $4a^2 - 4ax - 24x^2$.

31. $4a^2x^2 - 4ax^3 - 4x^4 + 4ax^2y + 4x^2y^2 - a^2y^2 - 2ay^3 - y^4 + x^4$.

32. $ab - ay + bx + az - xy + y^2 + xz - by - yz$.

33. $a^2 + 3ab + 2b^2$.

158. By multiplying $(x + a_1)(x + a_2) \ldots\ldots (x + a_n)$, and collecting the terms containing the like powers of x, we shall find,

1st. The highest power of x will be x^n, and its coefficient will be 1.

2d. The coefficient of x^{n-1} will be $a_1 + a_2 + a_3 \ldots\ldots a_n$.

3d. The coefficient of x^0 will be $a_1 a_2 \ldots\ldots a_n$.

Let the student illustrate this by multiplying five or six such factors.

159. From this we may often discover the binomial factors of a polynomial function of a single letter.

34. Required to factor $x^2 - 2x - 15$.

SOLUTION.—The factors of this will have numerical terms whose product is -15, and whose sum is -2, and we find that 3 and -5 fulfill these conditions. The factors, therefore, are $x + 3$ and $x - 5$.

35. Factor $3a^3 - 9a^2 - 12a + 36$.

SOLUTION.—Dividing by 3 to make the coefficient of a^3 unity, we have, $a^3 - 3a^2 - 4a + 12$. The product of the numerical terms of the factors is 12, their sum is -3, and there are *three* of these factors. The only three factors whose product is $+12$ and sum -3 are $+2$, -2, and -3. These give $a + 2$, $a - 2$, and $a - 3$, which with the 3 already taken out, are the factors sought.

Find the factors of the following:

36. $x^2 - 2x - 35$.

37. $x^2 + x - 20$.

38. $x^2 - 9x + 20$.

39. $4x^2 + 32x + 60$.

40. $5x^2 + 15x - 140$.

41. $x^3 - x^2 - 14x + 24$.

42. $x^3 - 8x^2 + 11x + 20$.

43. $5x^4 - 35x^3 + 85x^2 - 85x + 30$.

44. $2ax^3 + 2ax^2 - 32ax + 40a$.

45. $x^5 + x^4 - 5x^3 - 5x^2 + 4x + 4$.

CHAPTER VI.

DIVISORS AND MULTIPLES.

160. The ***Factors*** of a quantity are sometimes called its ***Divisors,*** since it may be divided by any one of them.

161. ***Commensurable Quantities*** have a *common divisor;* as, ac and ax, both of which have the *divisor* or *factor* a.

162. ***Incommensurable Quantities*** have *no common divisor;* as, abc and xyz. Such quantities cannot be measured with the *same unit;* hence the name. (Art. 8.)

GREATEST COMMON DIVISOR.

163. The ***Greatest Common Divisor*** of two or more quantities is that *common divisor* which contains the *greatest number of factors.*

164. If the *prime factors* of the quantities can be discovered by inspection, the ***g. c. d.*** may be found by combining all the common factors into a product, using each as many times as it is found in *every one of the quantities.*

165. When these factors cannot be thus discovered, we may employ a process based on the following

THEOREM.—*A factor common to two quantities is a factor of the remainder resulting from the division of one of the quantities by the other.*

DEMONSTRATION.—Let a and b represent the quantities, and let q and r be the quotient and remainder obtained by dividing a by b. Then will $a = bq+r$ or $a-bq = r$. Now every factor common to a and b is a factor of $a-bq$, and therefore of its equal r.

COR.—The ***g. c. d.*** of b and r will be the ***g. c. d.*** of a and b.

For, since $a = bq + r$, every divisor of a and b will be a divisor of b and r, and every divisor of b and r will divide a and b. Hence,

166. To find the ***g. c. d.*** of two quantities we have this

RULE.—I. *Find by inspection as many common factors of the two quantities as possible, and removing them from the quantities, reserve them as factors of the* ***g. c. d.***

II. *Reject also from each quantity all prime factors not common to both.*

III. *Divide one of the quantities remaining after these factors are removed by the other, and if there be a remainder, divide the divisor by the remainder, and the second divisor by the second remainder, and so on, until a divisor be found which gives no remainder. The product of this divisor with the common factors already removed will give the greatest common divisor.*

NOTES.—1. If, in the course of the operation, common factors are discovered, they should be removed, and reserved for the ***g. c. d.***

2. If *prime factors not common* are discovered, either in the *dividend* or *divisor*, they should be rejected before dividing.

167. If any division gives a *fractional term* in the quotient, thus involving fractional products, this may be avoided by multiplying that dividend by such a number as will make its first term divisible by the first term of the divisor. This will not affect the ***g. c. d.*** unless it introduce a new common factor. This it will not do, if the second part of the rule has been followed.

NOTE.—The ***g. c. d.*** will have the double sign ($\pm$); for, if $+a$ be a divisor of a quantity, $-a$ will also be a divisor. Hence, positive or negative factors may be *rejected* or *introduced.*

168. The Greatest Common Divisor of Polynomials.

The *greatest common divisor* of polynomials may be found by the *preceding rule,* as illustrated in the following examples:

Find the *g. c. d.* of the following polynomials:

1. $12x^5 - 51x^3 + 12x$ and $2x^5 - 4x^4 - 2x^3 + 4x^2$.

Analysis. $3x$ is evidently a factor of the first, and $2x^2$ of the second. The factor x being common to both, must be reserved as a factor of the greatest common denominator. but 3 and $2x$ may be rejected. We therefore divide the first by $3x$ and the second by $2x^2$, and proceed in accordance with the rule, as follows:

OPERATION.

$$\begin{array}{l|l}
4x^4 + 0 - 17x^2 + 0 + 4 & x^3 - 2x^2 - x + 2 \\ \hline
\underline{4x^4 - 8x^3 - 4x^2 + 8x} & 4x + 8 \\
\quad 8x^3 - 13x^2 - 8x + 4 & \\
\quad \underline{8x^3 - 16x^2 - 8x + 16} & \\
\qquad + 3x^2 \qquad - 12 &
\end{array}$$

Rejecting the factor 3 from the remainder, and dividing divisor by remainder,

$$\begin{array}{l|l}
x^3 - 2x^2 - x + 2 & x^2 - 4 \\ \hline
\underline{x^3 \qquad - 4x} & x - 2 \\
\quad - 2x^2 + 3x + 2 & \\
\quad \underline{- 2x^2 \qquad + 8} & \\
\qquad 3x - 6 &
\end{array}$$

Rejecting the factor 3, we have the remainder $x - 2$, which will of course divide $x^2 - 4$. (Art. 155.) Therefore the *g. c. d.* is

$$\pm (x - 2)\, x, \quad \text{or} \quad \pm (x^2 - 2x).$$

2. $4x^3 - 6x^2 - 4x + 3$ and $2x^3 + x^2 + x - 1$.

The operation may be put in the following convenient form.

OPERATION.

1st dividend,	$4x^3 - 6x^2 - 4x + 3$	$2x^3 + x^2 + x - 1$	1st divisor.
	$4x^3 + 2x^2 + 2x - 2$	2	1st quotient.
2d divisor,	$- 8x^2 - 6x + 5$	$8x^3 + 4x^2 + 4x - 4$	2d dividend.
2d quotient,	$- x$	$8x^3 + 6x^2 - 5x$	
3d dividend,	$- 8x^2 - 6x + 5$	$- 2x^2 + 9x - 4$	3d divisor.
	$- 8x^2 + 36x - 16$	4	3d quotient.
	$- 21\,)\, - 42x + 21$		
4th divisor,	$2x - 1$	$- 2x^2 + 9x - 4$	4th dividend.
	$- x + 4$	$- 2x^2 + x$	
		$+ 8x - 4$	
$\therefore \ \pm (2x - 1) =$ *g. c. d.* *Ans.*		$8x - 4$	

3. $ax^4 + 2a^2x^3 + a^3x^2 - ax^2 - 2a^2x - a^3$ and $a^6 - 2a^4x^2 + a^2x^4$.
4. $x^4 + 10x^3 + 24x^2 - 10x - 25$ and $4x^4 - 21x^2 + 5$.
5. $4a^2b^3 + a^3b + 2a^3b^2 + 2ab^4 + a^2b^3 - b^4$ and $a^3b - a^2b^2 - ab^3 + b^4$.
6. $(a^4 - b^4)\,ax$ and $(a^3 + b^3)\,bx$.
7. $(a^3 - b^3)(a - x)$ and $(a^2 - b^2)(a + x)$.
8. $3x^5 + 7x^3 - 5x^2 + 3$ and $3x^3 - 2x^2 - 1$.
9. $a^2 - 5ax + 4x^2$ and $a^3 - a^2x + 3ax^2 - 3x^3$.
10. $x^4 - 6x^3 + 13x^2 - 12x + 4$ and $x^3 - 4x^2 + 5x - 2$.

169. The ***g. c. d.*** of *more than two* quantities may be found by finding the ***g. c. d.*** of two, and then of this ***g. c. d.*** and a third, and so on till all are used.

Find the ***g. c. d.*** of the following:

11. $x^2 - x - 6$; $x^2 + 4x + 4$; $x^2 - 4$.
12. $3x^2 + 6x + 3$; $6x^2 - 30x - 36$; $9x^2 + 27x + 18$; $12x^2 - 12$.
13. $x^3 + 4x^2 + 6x + 9$; $x^3 + x^2 - 2x + 12$; $x^2 - x - 12$.
14. $x^4 - x^3 - 4x^2 + 16x - 24$; $x^3 - 5x^2 + 8x - 4$; $x^2 - 2x - 8$.
15. $x^4 - 8x^2 + 16$; $x^3 + 2x^2 - 4x - 8$; $x^3 - 2x^2 - 4x + 8$.

LEAST COMMON MULTIPLE.

170. A ***Multiple*** of a quantity is the *product* of that quantity by *any factor*. Hence,

It is any quantity which is divisible by the given quantity.

A ***Common Multiple*** is a multiple of *several quantities*.

The ***Least Common Multiple*** is the quantity which contains no factors except those which are necessary to make it a multiple of the several quantities.

The ***l. c. m.*** will therefore contain every factor found in the given quantities, and each factor will be found in the ***l. c. m.*** as many times as it is found in any one of the given quantities. Hence,

171. To find the ***l. c. m.*** of several quantities we have this

RULE.—I. *Separate the quantities into their prime factors.*

II. *Give each of these factors an exponent equal to the largest exponent it has in any of the given quantities.*

III. *The product of the quantities thus obtained will be the least common multiple sought.*

172. When the least common multiple of *two quantities* is required, it is evidently equal to

The product of the quantities divided by their ***g. c. d.*** Or,

One of the quantities divided by their ***g. c. d.*** *and multiplied by the other.*

For in the product of two quantities the *g. c. d.* will be found twice, and it is only necessary that it should be found once.

173. Find the ***l. c. m.*** of the following:

1. $x^3 - a^3$; $x^2 - a^2$; $x + a$; $x - a$.
2. $3a^2x^3y$; $6a^3x^2y^2$; $2axy^3$.
3. $x^2 + 2ax + a^2$; $a + x$; $a - x$.
4. $x^2 - 1$; $x + 1$; $x - 1$; $x^2 + 2x + 1$.
5. $x^3 - 1$; $x^2 + x - 2$; $x - 1$.
6. $x^n y^m$; $x^{m+n} y^{m+n}$; $x^m y^n$.
7. $(x^2 - 1)^2$; $(x - 1)^2$; $x + 1$.
8. $a^3 + x^3$; $a^2 - x^2$; $a^2 - ax + x^2$.
9. $a^3 - x^3$; $a^2 - x^2$; $a^2 + ax + x^2$.
10. $a^2 - x^2$; $a^2 - 2ax - x^2$; $a^2 + 2ax + x^2$.
11. $a^4 - 1$; $a^3 + a^2 + a + 1$; $a^3 - a^2 + a - 1$; $a^2 + 1$.
12. $x^6 - y^6$; $x^4 + x^2y^2 + y^4$; $x^3 + y^3$; $x^2 + y^2$.
13. $x^6 + a^6$; $x^4 - a^4$; $x^2 + a^2$; $x + a$.
14. $x^3 + x^2 - 10x + 8$; $x^2 + 2x - 8$; $x^2 - 3x + 2$; $x^2 - 1$.
15. $x^4 + 5x^3 + 5x^2 - 5x - 6$; $x^3 + 6x^2 + 11x + 6$; $x^3 + 4x^2 + x - 6$.

CHAPTER VII.

FRACTIONS.

174. ***Fractions*** in Algebra are *expressions for division*, in which the dividend is written over the divisor with *a line* between them, called the *dividing line*.

175. The ***Numerator*** is the quantity *above* the line, or the *dividend*.

176. The ***Denominator*** is the quantity *below* the line, or the *divisor*.

177. The ***Terms*** of a fraction are its *numerator* and *denominator*.

178. A fraction is in its ***lowest terms*** when the numerator and denominator have *no common factor*.

179. The ***Value*** of a fraction is the *quotient* of the *numerator* divided by the *denominator*.

180. An ***Integral*** or ***Entire Quantity*** is one expressed without fractions.

181. A ***Mixed Quantity*** is one which has an *integral* and a *fractional* part.

182. Fractions whose denominators are alike are said to have a ***Common Denominator***.

183. A ***Complex Fraction*** is one having a fraction in the numerator or denominator, or both.

NOTE.—The terms *proper*, *improper*, *simple*, and *compound*, when applied to algebraic fractions, have the same meaning as in Arithmetic.

SIGNS OF FRACTIONS.

184. ***Every Fraction*** has the sign $+$ or $-$ expressed or understood before the *dividing line;* that is, when its direction is considered. (Art. 86, 1°.)

185. The ***Dividing Line*** has the force of *a vinculum* or *parenthesis*, and the sign before it belongs to the value of the fraction.

186. ***Every Numerator and Denominator*** is preceded by the sign $+$ or $-$, expressed or understood. In this case, the sign affects only the single term to which it is prefixed.

187. If *the sign before the dividing line is changed,* the sign of the fraction is changed.

Thus, $a + \frac{bx}{x} = a + b$, but $a - \frac{bx}{x} = a - b$.

188. If *all the signs of the numerator are changed,* the sign of the fraction is changed.

Thus, $\frac{+ax}{x} = +a$, and $\frac{-ax}{x} = -a$.

189. If *all the signs of the denominator are changed,* the sign is also changed.

Thus, $\frac{ax}{+x} = +a$; but $\frac{ax}{-x} = -a$. Hence,

190. If *any two of these changes* are made at the same time, the sign of the fraction will not be changed.

Thus, $\frac{ax}{x} = +a$. Changing the signs of both numerator and denominator, $\frac{-ax}{-x} = +a$.

NOTE.—By the application of the above principles, the *sign* of either term, or that before the dividing line, may be made plus, if desirable.

PRINCIPLES.

191. The principles for the treatment of fractions in Algebra are the same as those in Arithmetic.

1°. *Multiplying the numerator,* or *Dividing the denominator,*	*Multiplies the fraction.*

Thus, $\frac{2\times 2}{6}=\frac{4}{6}=\frac{2}{3}$. And $\frac{2}{6\div 2}=\frac{2}{3}$.

2°. *Dividing the numerator,* or *Multiplying the denominator,*	*Divides the fraction.*

Thus, $\frac{2\div 2}{6}=\frac{1}{6}$. And $\frac{2}{6\times 2}=\frac{2}{12}=\frac{1}{6}$.

3°. *Multiplying* or *dividing both terms by the same quantity*	*Does not change its value.*

Thus, $\frac{2\times 2}{6\times 2}=\frac{4}{12}=\frac{2}{6}=\frac{1}{3}$. And $\frac{2\div 2}{6\div 2}=\frac{1}{3}$.

REDUCTION OF FRACTIONS.

192. ***Reduction of Fractions*** is changing their *terms* without altering the *value* of the fractions.

CASE I.

193. To Reduce a Fraction to its Lowest Terms.

1. Reduce $\frac{5a^2b^2dx}{15abcx}$ to its lowest terms.

ANALYSIS.—By inspection, we perceive the factors 5, a, b, and x are common to both terms. Cancelling these common factors, the fraction becomes $\frac{abd}{3c}$. Now since both terms have been divided by the same quantity, the value of the fraction is not changed. (Art. 191, 3°.) And since these terms have no common factor, it follows that $\frac{abd}{3c}$ are the lowest terms required. (Art. 178.)

NOTE.—It will be observed that the factors 5, a, b, and x are prime; therefore, the product $5abx$ is the *g. c. d.* of the numerator and denominator. (Art. 164.) Hence, the

RULE.—*Cancel all the factors common to the numerator and denominator.*

Or, *Divide both terms of the fraction by their greatest common divisor.* (Art. 178.)

Reduce the following fractions to their lowest terms:

2. $\dfrac{27a^3b^2c^5d}{162a^5c^4d^3e}$.

3. $\dfrac{6a^{\frac{1}{2}}b^{\frac{1}{3}}c^{\frac{1}{4}}}{9a^{\frac{3}{2}}b^{\frac{5}{3}}c^{\frac{7}{4}}}$.

4. $\dfrac{x^3+y^3}{x^2-y^2}$.

5. $\dfrac{x^6+1}{x^4-1}$.

6. $\dfrac{x^3+y^3}{x^6-y^6}$.

7. $\dfrac{x^2-x-42}{x^2+9x+18}$.

8. $\dfrac{4x^2-4}{6x^4-6}$.

9. $\dfrac{3a^2-6ax+3x^2}{3x^2-3a^2}$.

CASE II.

194. To Reduce a Fraction to an Entire or Mixed Quantity.

Since a fraction is only an expression for division, we have the

RULE.—*Divide the numerator by the denominator.*

Reduce the following to entire or mixed quantities:

1. $\dfrac{4x^3y^{\frac{4}{5}}z^3}{2x^2y^{\frac{1}{5}}z^2}$.

2. $\dfrac{b^3-a^2b^2}{1-a^2}$.

3. $\dfrac{a^3-b^3}{a-b}$.

4. $\dfrac{a^{10}+b^{10}}{a^2+b^2}$.

5. $\dfrac{a^{18}+b^{18}}{a^6+b^6}$.

6. $\dfrac{a^3x^4+2a^2x^3+ax^2}{a^2x^2+ax}$.

7. $\dfrac{x^3+2x^2-12x-13}{x^2+x-12}$.

8. $\dfrac{a^2b^2-b^2+ab-b-a+1}{a^2-1}$.

CASE III.

195. To Reduce a Mixed Quantity to an Equivalent Fraction.

1. Reduce $a + \frac{b}{3}$ to the form of a fraction.

ANALYSIS.—Since in 1 unit there are three thirds, in a units there must be a times 3 thirds, or $\frac{3a}{3}$; and $\frac{3a}{3} + \frac{b}{3} = \frac{3a + b}{3}$, the fraction required. Hence, the

OPERATION.

$$a + \frac{b}{3} = \frac{3a}{3} + \frac{b}{3}$$

$$\frac{3a}{3} + \frac{b}{3} = \frac{3a + b}{3}, \textit{ Ans.}$$

RULE.—*Multiply the integral part by the denominator; to the product add the numerator, and place the sum over the denominator.*

NOTE.—An entire quantity may be reduced to the form of a fraction by making 1 its denominator. Thus, $a = \frac{a}{1}$.

Reduce the following to the fractional form:

2. $ax^2 - \frac{b^3 + a^2x^2}{a}$.

3. $ab + \frac{x - a^2b^2}{ab}$.

4. $x + 1 + \frac{(x - 1)^2}{x + 1}$.

5. $a - x - \frac{(a + x)^2}{a - x}$.

6. $a + b - \frac{a^2 + b^2}{a + b}$.

7. $x^3 + x^2 + x + 1 - \frac{x^4 - 1}{x - 1}$.

CASE IV.

196. To Reduce a Fraction to any required Denominator.

By Art. 191, 3°, we have the

RULE.—*Multiply both terms by the factor which will give the fraction the required denominator.*

NOTES.—1. An entire quantity may be reduced to a fraction having a given denominator, by the same rule, by first writing under it the denominator 1.

2. When the required denominator is not a multiple of the given denominator, the result will be a complex fraction.

EXAMPLES.

1. Reduce $\frac{a^2+2}{c}$ to a fraction whose denominator is $a^2c - c$.

2. Reduce $\frac{a^2-1}{a^2+1}$ to a fraction whose denominator is a^4+1.

3. Reduce $\frac{a+1}{a-1}$ to a fraction whose denominator is a^4-1.

4. Reduce $\frac{x+5}{x-7}$ to a fraction whose denominator is $x^2-2x-35$.

5. Reduce $\frac{a}{a^2-1}$ to a fraction whose denominator is a^3-1.

6. Reduce $x - \frac{-1}{x+1}$ to a fraction whose denominator is x^2-1.

CASE V.

197. To Reduce Fractions to a Common Denominator.

By Case IV we can reduce fractions to any required denominator; but to avoid *complex fractions* that denominator must be a *multiple* of the given denominators.

To express the fractions in the *lowest terms,* it must be the *least common multiple.* We have then the following

RULE.—I. *Find a common multiple of the denominators for the common denominator, the least common multiple being preferred.*

II. *Multiply both terms of each fraction by that factor which will give it the required denominator.*

198. When the least common multiple is taken as a common denominator, it is called the ***Least Common Denominator.***

Reduce the following fractions to the least common denominator:

1. $\dfrac{x}{x+y}, \quad \dfrac{x^2}{x^2-y^2}, \quad \dfrac{x^3}{x^4-y^4}.$

2. $\dfrac{1}{x^4-1}, \quad \dfrac{2}{x^4+1}, \quad \dfrac{3}{x^2+1}.$

3. $\dfrac{a+1}{2a^2+2}, \quad \dfrac{a-1}{4a^2-4}, \quad \dfrac{a^2-1}{8a^6-8}.$

4. $\dfrac{4}{x^2-1}, \quad \dfrac{x-2}{x^2+x-2}, \quad \dfrac{x+2}{x^2-x-2}.$

5. $\dfrac{x}{x^4-1}, \quad \dfrac{x^2+1}{x^4+4x^2+3}, \quad \dfrac{x^2-1}{x^4+2x^2-3}.$

6. $\dfrac{x+y}{x^4-y^6}, \quad \dfrac{x-y}{x^2+y^3}, \quad \dfrac{x^2+y^2}{x^3-y^3}.$

ADDITION AND SUBTRACTION OF FRACTIONS.

199. According to Art. 113, fractions may be added or subtracted by the following

RULE.—*Reduce the fractions to a common denominator, and add or subtract their numerators, writing the result over the common denominator.*

1. Add $\dfrac{2ax}{3b}$ and $\dfrac{3a}{2x}$.

SOLUTION.—Reduced to a *c. d.*, the fractions become $\dfrac{4ax^2}{6bx}$ and $\dfrac{9ab}{6bx}$, which, by adding the numerators, give $\dfrac{4ax^2+9ab}{6bx}$, *Ans.*

2. From $\dfrac{7ab}{x}$ take $\dfrac{5cd}{y}$.

SOLUTION.—Reducing to a *c. d.*, we have $\dfrac{7aby}{xy}$ and $\dfrac{5cdx}{xy}$; and $\dfrac{7aby}{xy}-\dfrac{5cdx}{xy}=\dfrac{7aby-5cdx}{xy}$, *Ans.*

Perform the operations indicated in the following:

3. $\frac{2ax}{b^2} + \frac{a^2 + x^2}{b^2} - \frac{x^3 - 2a^3}{ab^2 + b^2x}$.

4. $\frac{2ab}{a^2 - b^2} + \frac{a - b}{a + b} + \frac{a + b}{a - b}$.

5. $\frac{x - y}{x + y} + \frac{4xy}{x^2 - y^2} - \frac{x + y}{x - y}$.

6. $\frac{2}{x + 1} - \frac{x + 1}{x - 1} - \frac{-4}{x^2 - 1}$.

7. $\frac{1}{x + 3} - \frac{-7}{x^2 - x - 12} + \frac{x + 3}{x - 4}$.

8. $\frac{a^2 + x^2}{x + a^2} - \frac{x^3 + a^4x}{x^2 - a^4} - \frac{-a^2x}{x - a^2}$.

9. $\frac{x}{x^2 - 9} - \frac{x}{x^2 - x - 6} + \frac{1}{x^2 + 5x + 6}$.

10. $\frac{-1}{x - 1} - \frac{-1}{x + 1} - \frac{-1}{x^2 - 1} - \frac{-1}{x^3 - 1}$.

200. The *sum* of the numerators of several *equal* fractions, placed over the sum of the denominators, forms a fraction *equal* to one of the given fractions; for, if they be reduced to a *c. d.*, they will be *identical*, and the operation is equivalent to multiplying both terms of one fraction by the number of fractions.

MULTIPLICATION OF FRACTIONS.

CASE I.

201. To Find the Product of an Integral Quantity and a Fraction.

According to Art. 191, 1°, we have the following

RULE.—*Cancel all factors common to the integral quantity and the denominator, and multiply the numerator by the remaining factors of the integral quantity.*

NOTES.—1. A fraction is multiplied by any factor by *cancelling that factor* from its *denominator.*

2. Cancelling the *whole denominator multiplies the fraction by the denominator.*

Find the product of the following:

1. $\dfrac{x}{a^2 - x^2} \times (a - x).$

2. $\dfrac{1}{x^4 - 1} \times (x^2 + 1)(x - 1).$

3. $(x^4 + x^3 + x^2 + x + 1) \times \dfrac{-1}{x^5 - 1}.$

4. $(x^3 + 1)(x^3 - 1) \times \dfrac{5}{x^6 - 1}.$

5. $\dfrac{x + 4}{x^2 + 4x - 21} \times (x + 7).$

6. $(x^2 + 5x + 6) \times \dfrac{x}{x^3 + 9x^2 + 26x + 24}.$

7. $(x^2 + a^5) \times \dfrac{1}{x^4 - a^{10}}.$

8. $\dfrac{10}{x^{12} + a^3} \times (x^4 + a).$

CASE II.

202. To Multiply a Fraction by a Fraction.

1. Multiply $\dfrac{a}{b}$ by $\dfrac{m}{n}$.

SOLUTION.—By Arts. 53 and 54, the product may be written

$$\frac{a}{b} \times \frac{m}{n}, \text{ or } ab^{-1} \times mn^{-1}, \text{ or } ab^{-1}mn^{-1}, \text{ or } \frac{am}{bn}.$$

If in this result there are *common factors* in the *numerator* and *denominator*, they might have been *cancelled* before multiplying. Hence, the

RULE.—I. *Cancel all factors of the numerators and denominators common to both.*

II. *Multiply the remaining factors of the numerators for the numerator of the product, and the remaining factors of the denominators for the denominator of the product.*

NOTE.—This rule applies to any number of factors.

Find the product of the following:

1. $\dfrac{a}{a^2 - x^2} \times \dfrac{a + x}{a}$.

2. $\dfrac{x^2}{a^2 - b^4} \times \dfrac{a - b^2}{x^2}$.

3. $\dfrac{1}{x^2 + x + 1} \times \dfrac{3}{x - 1}$.

4. $\dfrac{1}{a^4 - b^6} \times \dfrac{a^2 + b^3}{a^2 - b^3}$.

5. $\dfrac{x^2 + a^3}{x^3 + a^2} \times \dfrac{x^9 + a^6}{x^6 + a^9}$.

6. $\dfrac{x^3 - a^5}{x^5 - a^3} \times \dfrac{x^{10} - a^6}{x^6 - a^{10}}$.

7. $\dfrac{a^2 + a - 42}{x^2 + x - 30} \times \dfrac{x^2 - x - 20}{a^2 - a - 30}$.

8. $\dfrac{a + 1}{x^3 + 1} \times \dfrac{x^3 - 1}{a^3 + 1} \times \dfrac{a - 1}{x - 1}$.

9. $\dfrac{x}{x^2 + 1} \times \dfrac{x^4 - 1}{x^2} \times \dfrac{x^2}{x - 1} \times \dfrac{x^4 + x^2 + 1}{-x}$.

DIVISION OF FRACTIONS.

CASE I.

203. To Divide a Fraction by an Integral Quantity.

By Art. 188, 2°, we have the

RULE.—*Cancel from the numerator of the fraction and the divisor all common factors, and multiply the denominator by the remaining factors of the divisor.*

Divide the following:

1. $\dfrac{a^2bc^{\frac{1}{2}}}{3x^{\frac{1}{3}}yz^2} \div a^2c^{\frac{1}{3}}x^{\frac{1}{3}}$.

2. $\dfrac{8x^{\frac{2}{3}}y^{\frac{1}{3}}z^2}{9a^{\frac{1}{2}}c^{\frac{1}{3}}d} \div 4a^{\frac{2}{3}}c^{\frac{2}{3}}x^{\frac{1}{4}}y^{\frac{1}{3}}$.

3. $\dfrac{x^3 + 27}{x - 2} \div (x^2 + 5x + 6)$.

4. $\dfrac{a^2x^3 - a^4x}{c} \div [a^2 (x + a) (x - a)]$.

5. $\dfrac{8 + 6a + a^2}{3 + 4a + a^2} \div (6 - 5a - 2a^2 + a^3)$.

6. $\dfrac{a^3 - y^3}{x + y} \div [ax - (a + x) y + y^2]$.

CASE II.

204. To Divide a Fraction by a Fraction.

1. Divide $\frac{a}{b}$ by $\frac{m}{n}$.

SOLUTION.—By Art. 54 this may be written

$$\frac{a}{b} \div \frac{m}{n}, \quad \text{or} \quad ab^{-1} \div mn^{-1}, \quad \text{or} \quad \frac{ab^{-1}}{mn^{-1}}, \quad \text{or} \quad \frac{am^{-1}}{bn^{-1}}, \quad \text{or} \quad \frac{an}{bm}.$$

From the last two expressions we derive the

RULE.—*Divide the terms of the dividend by the corresponding terms of the divisor.*

Or, *Multiply the dividend by the divisor inverted.*

These operations may be combined in the following

RULE.—*Cancel all factors common to both numerators, also those common to both denominators, and multiply the dividend by the divisor inverted.*

205. To Divide an Integral Quantity by a Fraction.

Make the *integer* a *fraction* by writing 1 for a denominator, and proceed as above.

1. Divide $(5a - b)$ by $\frac{b}{5a}$.

2. Divide $(1 - x^2)$ by $\frac{1 + x}{3a}$.

206. ***Complex Fractions*** are reduced to *simple ones* by performing the division indicated.

Divide the following:

1. $\dfrac{2a^2x^{\frac{4}{3}}y}{6b^3c^{\frac{1}{4}}d^2} \div \dfrac{ax^{\frac{1}{3}}y^2}{b^2c^{\frac{5}{4}}d^2e}$.

2. $\dfrac{c^{15} - x^{18}}{a^6 + x^6} \div \dfrac{c^5 - x^6}{a^2 + x^2}$.

3. $\dfrac{c^6 - d^6}{a^2b^4 - c^4d^2} \div \dfrac{ac^3 - ad^3}{ab^4 - b^2c^2d}$.

4. $\dfrac{y^{12} + 1}{a^6 - 1} \div \dfrac{y^4 + 1}{a^3 + a^2 - a - 1}$.

5. $\dfrac{x^{14}+y^{-14}}{a^{-7}+y^{-7}} \div \dfrac{a^2x^2+a^2y^{-2}}{a^{-1}+y^{-1}}$.

6. $\dfrac{c^2-5c-14}{c^2-10c-39} \div \dfrac{c^2+11c-26}{c^2+c-6}$.

7. $\dfrac{\dfrac{a^2-4}{a^2-16}}{\dfrac{a^2-2a-8}{a^2+2a-8}}$.

8. $\dfrac{\dfrac{a^9-y^{21}}{a^{16}-y^{20}}}{\dfrac{a^3-y^7}{(a^8+y^{10})(a^4+y^5)}}$.

207. The various rules for multiplication and division of integers and fractions, whatever may be the number of multipliers and divisors, may be summed up in the following

GENERAL RULE.

I. *Reduce integers and mixed quantities to fractions.*

II. *Invert all the divisors.*

III. *Cancel all the factors common to the numerators and denominators.*

IV. *Combine the remaining factors for the result.*

208. From the nature of fractions we readily see that the value of a fraction may be any quantity whatever; for as the denominator *decreases,* with a given numerator, the value of the fraction *increases,* and when the denominator becomes infinitely small, or infinitesimal (a quantity represented by o), the value of the fraction becomes *infinite.*

209. On the other hand, when with a given denominator the numerator *decreases indefinitely,* the value of the fraction decreases and becomes *infinitesimal* or o. Hence we have the equations,

$$\frac{a}{0} = \infty. \quad (1) \qquad\qquad \frac{a}{\infty} = 0. \quad (3)$$

$$\frac{0}{a} = 0. \quad (2) \qquad\qquad \frac{0}{0} = a. \quad (4)$$

210. These are important relations for the student to remember. Translated into common language they become:

1st. *A finite quantity divided by zero equals infinity.*

2d. *Zero divided by a finite quantity equals zero.*

3d. *A finite quantity divided by infinity equals zero.*

4th. *Zero divided by zero equals any quantity.*

Equation (4) is called *indeterminate* because, as it stands, its value cannot be determined. It is only when the functions are known from which the two zeros have resulted that a *definite value* can be given to such a fraction, and then only when these functions are dependent. For example, if $\frac{a-x}{b-y}$ becomes $\frac{0}{0}$ by reason of x becoming equal to a and y to b, the resulting $\frac{0}{0}$ is indeterminate, and such an answer to a problem must be interpreted as meaning that the problem has no definite answer, but its conditions are satisfied by any value whatever for the required quantity.

But if $\frac{a(a^2-x^2)}{x(a-x)}$ become $\frac{0}{0}$ by reason of x becoming equal to a, we may find a definite value for the expression by first reducing the fraction to its lowest terms, thus,

$$\frac{a(a^2-x^2)}{x(a-x)}=\frac{a(a+x)}{x},$$

which when $x=a$, becomes $2a$.

Perform the operations indicated in the following, and simplify the expressions:

1. $\frac{a+b}{a-b}+\frac{a-b}{a+b}-\frac{a^2-b^2}{a(a-b)}.$

2. $\frac{a-1}{a}+\frac{a}{a-1}.$

3. $\frac{a^3-x^3}{a^2-x^2}-\frac{a^2-x^2}{a-x}.$

4. $\frac{a-1}{a+1}+\frac{a+1}{a-1}-\frac{a^2-1}{a^2+1}.$

5. $\frac{a-1}{a+1}\times\frac{a+1}{a-1}\div\frac{a^2-1}{a+1}.$

6. $\frac{x^3-a^3}{x^2-a^2}\div\frac{x-a}{x+a}.$

7. $a + x - \dfrac{a^2 + x^2}{a + x}$.

8. $\dfrac{a - x}{ax} + \dfrac{b - x}{ab} + \dfrac{x - b}{bx}$.

9. $\dfrac{1}{x - 1} + \dfrac{1}{x + 1} - \dfrac{x - 1}{x^2 - 1}$.

10. $\left\{\dfrac{\dfrac{a^4 - x^4}{x^2 - y^2} \div \dfrac{a^2 + x^2}{x + y}}{\dfrac{1}{a + x} - \dfrac{1}{a - x}} \div \dfrac{x^2 - a^2}{x - y}\right\} 2x$.

11. $\dfrac{a^3 + 3a^2x + 3ax^2 + x^3}{a^2 + 2ax + x^2}$.

12. $\dfrac{\left(a - \dfrac{1}{a}\right)\left(b - \dfrac{1}{b}\right)}{\left(\dfrac{a}{b} - 1\right)\left(\dfrac{b}{a} + 1\right)}$.

13. $\left(\dfrac{a}{a + b} + \dfrac{b}{a - b}\right) \div \left(\dfrac{a}{a - b} - \dfrac{b}{a + b}\right)$.

14. $\left(x^2 + \dfrac{1}{x^2} - 2\right) \div \left(x - \dfrac{1}{x}\right)$.

15. $\dfrac{\left(\dfrac{a^2 + b^2}{b} - a\right)\dfrac{a^2 - b^2}{a^3 + b^3}}{\dfrac{1}{b} - \dfrac{1}{a}}$.

16. $\dfrac{1}{x - \dfrac{1}{x + \dfrac{1}{x}}} \div \dfrac{1}{x}$.

17. $(a^3 - 3a^2 + 3a + 1) \div \left(a^2 - 2a + \dfrac{a + 1}{a - 1}\right)$.

18. $\dfrac{\dfrac{x}{y} + \dfrac{y}{x}}{\dfrac{y}{x} - \dfrac{x}{y}} \div \dfrac{(x + y)^2 - 2xy}{x - y}$.

CHAPTER VIII.

EQUATIONS OF THE FIRST DEGREE.

211. ***Algebra*** has already been defined as the ***Science of the Equation.*** The preceding chapters have been devoted to the fundamental operations upon quantity in preparation for the reduction and discussion of *equations.*

212. An ***Equation*** is an *expression* of *equality* between two quantities; as, $a = b + c$. (Art. 26.)

213. These two quantities are called ***Members of the Equation,*** the one on the left of the sign $=$ being the ***First Member,*** and the one on the *right* the ***Second Member.***

214. ***Equations*** are ***Literal*** when the unknown quantities have *literal coefficients,* and ***Numerical*** when their coefficients are *numerical.*

Thus, $ax^2 + bx = c$ is a *literal* equation, and $3x^2 - 2x = 10$ is a *numerical* equation.

215 Equations are of *different degrees,* depending on the *exponents* of the unknown quantity.

To determine the *degree,* these exponents must be expressed in the *same unit;* that is, must have a *common denominator.*

The *degree* will then be found by subtracting the *least* from the *greatest* exponent of the unknown quantity; or, if there are no negative exponents, the degree will be equal to the *greatest exponent.*

Thus, $x^2 - x^{-1} = 5$ is of the third degree; $x + \frac{1}{x} = 7$ is of the second degree; and $x^2 + x^{\frac{1}{2}} = 2$ is of the $\frac{4}{3}$ degree.

216. An ***Identical Equation*** is one whose members are *identical*, or may be made so without changing their value; as,

$$a + x = a + x;$$
$$(a + x)^2 = a^2 + 2ax + x^2.$$

217. Such an equation is called *absolute*, since its truth does not depend on the value given to any of the quantities.

218. A ***Conditional Equation*** is one which is true only on *certain conditions*; as,

$$x - 2 = 5,$$

which is true only when $x = 7$.

219. Such equations are used in the solution of problems. The conditions of the problem are expressed by the equation, in which some letter, as x, is put for the unknown quantity. Whatever value of x will make the equation true, will therefore satisfy the conditions of the problem.

220. A ***Root of a Conditional Equation*** is that value of the unknown quantity which will satisfy the equation; as,

$$x + 5 = 2x,$$

in which 5 is a *root*, since in substituting it for x we have

$$5 + 5 = 2 \times 5.$$

REDUCTION OF EQUATIONS.

221. The ***Reduction of an Equation*** consists in such transformations as will make the *unknown quantity* an *explicit* function of the *known quantities*. (Arts. 33, 34.)

Thus, in the equation,

$$\frac{a + x}{b} = \frac{a - x}{c},$$

x is a function of a, b, and c; that is, the value of x depends on the values given to a, b, and c. This is *implied* in the equation; for to affirm that any combination of x with a, b, and c is equal to a different combination of the same quantities is to make each one of the quantities depend on

the others for its value. If *all but one* have arbitrary values assigned them, the remaining one cannot have such a value assigned to it and make the equation true; but it must have a *definite value*, which will be expressed by some *combination* (that is, some *function*) of those quantities to which the arbitrary values have been assigned.

222. Those quantities to which arbitrary values may be assigned are called ***Known Quantities,*** and those which *are to be found* by their *relations* to the *known* are called ***Unknown.*** (Arts. 27, 28.)

223. In the equation above, if we let a, b, and c represent *known quantities,* the equation ***implies*** that x is a function of a, b, and c, but does not *explicitly* state *what function* it is. When the equation is reduced, it becomes

$$x = \frac{ab - ac}{b + c},$$

which states ***explicitly*** what function of a, b, and c, equals x.

224. This reduction is effected by the axiom,

Equal quantities equally affected remain equal. (Art. 38.)

225. The ***Method of Reduction*** depends on the *degree of the equation.* (Art. 215.)

226. ***Equations of the First Degree*** are also called ***Simple Equations,*** and can contain only two powers of the unknown quantity. These powers (unless there are negative exponents) will be the *first* and the *zero power.*

NOTE.—Let the student observe that the term which *does not contain* the *unknown quantity,* commonly called the *absolute term,* is said to contain the *zero power of that quantity.*

227. The ***Reduction*** of *equations* of the *first degree* is effected by the following operations:

1st. Clearing of fractions; that is, removing denominators.

2d. Removing known terms from the first member and unknown terms from the second member.

3d. Uniting similar terms.

4th. Making the coefficient of the unknown quantity unity.

NOTE.—These operations may be performed in any order which is most convenient.

228. To Clear an Equation of Fractions.

RULE.—*Multiply the equation by the least common multiple of its denominators.*

229. To Remove a Term from either Member.

RULE.—*Subtract the term from both members.*

230. To Make the Coefficient of the Unknown Quantity Unity.

RULE.—*Divide both members by this coefficient.*

EXAMPLES.

231. Find the value of x in the following equations:

1. $$\frac{3x + 2a}{2} - \frac{x - 5a}{3} = 5a.$$

OPERATION.—Multiplying by 6,

$$9x + 6a - 2x + 10a = 30a.$$

Subtracting $6a$ and $10a$ from both members, and uniting similar terms,

$$7x = 14a.$$

Dividing by 7, $x = 2a$, *Ans.*

NOTE.—It will be observed that the *removal* of a term from one member of an equation causes it to appear in the other member with an *opposite sign.*

This is called ***Transposition;*** because it is equivalent to *transposing* a term from one member to the other and *changing its sign.*

232. Many equations *not* of the *first degree* may be made such by some operation affecting equally both members. Thus,

2. Reduce $x^2 - 4 = 2(x + 2)$.

Dividing by $x + 2$, $x - 2 = 2.$

Transposing, $x = 4$, *Ans.*

This equation is of the second degree, but by dividing both members by $x + 2$ it is reduced to the first degree, and 4 found to be a root.

But since both members are divisible by $x + 2$, any value of x that will make this factor zero will satisfy the equation, for it will reduce both members to zero.

Thus, $x + 2 = 0$ gives $x = -2$.

$\therefore$ -2 is another root of the equation.

We shall see hereafter that every equation has as many *roots* as there are *units* in its degree, and that when we divide by a factor containing x, we must make another equation by putting that factor equal to zero, and find its roots, if we would find all the roots of the original equation.

3. Reduce $(x + 1)^{\frac{1}{2}} = -2$.

Squaring both members, $x + 1 = 4$.

Transposing, etc., $x = 3$.

233. To Prove the Correctness of the Work of Reduction.

Substitute the root found for the unknown quantity in the equation. If correct, it will reduce to an identical equation.

Thus, in the last example, substituting 3 for x, we have,

$$(3 + 1)^{\frac{1}{2}} = -2.$$

But $(3 + 1)^{\frac{1}{2}} = 4^{\frac{1}{2}} = +2$ or -2.

While, therefore, the equation is satisfied by the root 3 if the *negative* root of 4 be taken, it is not satisfied if the *positive* root be taken.

But the equation is of the $\frac{1}{2}$ degree, and if the statement in Art. 232 be true, the root found should be a *half-root*, as we see it is; that is, it satisfies the equation only in one-half the ways in which it can be substituted.

4. Reduce $\dfrac{x + a}{b} + \dfrac{x - a}{b} = 10$.

In this equation we may unite the terms with advantage before clearing of fractions; thus,

$$\frac{2x}{b} = 10.$$

Dividing by $\frac{2}{b}$, we have $x = 5b$, *Ans.*

5. Reduce $ax + bx = c$.

Dividing by $a + b$, $x = \dfrac{c}{a + b}$, *Ans.*

Reduce the following, and prove the work by substituting the root found:

6. $\frac{x}{2} + \frac{x}{3} - \frac{x}{4} = \frac{x+2}{2}.$

7. $\frac{x-1}{2} - \frac{x-3}{4} = \frac{x-3}{2}.$

8. $\frac{a-x}{a} - \frac{b-x}{b} = \frac{c-x}{c}.$

9. $ax^2 - bx = bx^2 - cx.$

10. $(a+x)(b+x) = (m+x)(n+x).$

11. $\frac{x-a}{b} + \frac{x-b}{c} + \frac{x-c}{a} = \frac{x-(a+b+c)}{abc}.$

12. $\frac{1}{x-2} - \frac{1}{x-4} = \frac{1}{x-6} - \frac{1}{x-8}.$

13. $\frac{m(x+a)}{x+b} + \frac{n(x+b)}{x+a} = m+n.$

14. $x + \frac{3x-5}{2} = 12 - \frac{2x-4}{3}.$

15. $3x - \frac{x-4}{4} - 4 = \frac{5x+14}{3} - \frac{1}{12}.$

16. $5 - 6x + \frac{7x+14}{3} = \frac{17-3x}{5} - \frac{4x+2}{3}.$

17. $\frac{20-x}{2} - \frac{6x-8}{7} + \frac{4x-4}{5} = x - \frac{3x-3}{5} + 4.$

18. $\frac{2x+4}{3} = \frac{6x+7}{9} + \frac{7x-13}{6x+3}.$

19. $4\left(\frac{x-b}{3}\right) = 3\left(\frac{x+b}{4}\right) + \frac{x-b}{3}.$

20. $\frac{x}{a+1} - b = \frac{x}{a-1}.$

21. $\frac{c}{a^2-b^2} = \frac{x}{a-b} - \frac{2+x}{a+b}.$

22. $\frac{x-2}{5} + \frac{.301}{.5} = .001x + .6 - \frac{x-2}{.05}.$

SOLUTION OF PROBLEMS.

234. The ***Solution of a Problem,*** which requires the finding of an unknown quantity from its relations to known quantities, consists of three distinct steps. (Art. 30.)

1st. The *conditions of the problem* must be expressed by a *conditional equation.*

2d. That equation must be *reduced* from an *implicit* to an *explicit* function, called a *formula.*

3d. The *numerical values* of the known quantities must be *substituted in that formula.*

235. The first of these steps does not belong to Algebra; but as the practical value of Algebra cannot be illustrated except by the solution of problems, it is important to become familiar with the translation of the conditions of a problem into an equation.

The problems, however, which may properly engage the attention of the student are those relating to subjects with which he is supposed to be familiar.

236. This analysis of the process of solving a problem gives the following

RULE.—I. *Represent the quantities involved by proper letters, in accordance with the usage of the algebraic language.* (Arts. 41–69.)

II. *With these letters express the conditions of the problem by a conditional equation.*

III. *Reduce this equation.* (Arts. 224–227.)

IV. *To apply the solution to a special case, substitute the numerical values given in the special case, in the formula obtained.*

237. The following problems will illustrate the rules:

PROBLEM 1. Find the number which being divided by two given numbers will give quotients differing by a given number.

SOLUTION.—1st. Let $x =$ the number to be found.
m and $n =$ the given divisors.
$d =$ the difference of quotients.

Having assumed this notation, we are prepared to express the conditions of the problem by an equation. This equation will be

$$\frac{x}{m} - \frac{x}{n} = d,$$

in which x is an *implicit* function of m, n, and d.

2d. The second step in the solution is the reduction of this equation, by which x becomes an *explicit function* of the given quantities. This reduction gives

$$x = \frac{mnd}{n - m}$$

3d. The third step consists in applying this *explicit function* or *formula* to any given case of the problem.

2. Find a quantity whose fifth part exceeds its sixth part by 5.

Here $m = 5$, $n = 6$, and $d = 5$; and

$$x = \frac{5 \cdot 6 \cdot 5}{6 - 5} = 150, \textit{ Ans.}$$

238. The third step of the solution, which is arithmetical, may be performed before the second by substituting in the equation, before it is reduced, the numbers belonging to the special case.

Thus, $$\frac{x}{5} - \frac{x}{6} = 5.$$

Reducing, $$x = 150, \textit{ Ans.}$$

239. The advantage of reducing the equation before the arithmetical substitutions are made is evident from the fact that a *formula* or *rule* is obtained by which the *arithmetical* part of the solution may be performed for any special case of the problem.

DISCUSSION OF FORMULAS.

240. A problem is said to be ***Generalized*** when its conditions are stated in *general terms* and reduced to a *formula*.

The ***Discussion of a Formula*** consists in applying it to such special cases of the problem as will show the different forms the result may take.

3. Find the time when the ages of two persons, A and B, will have a given ratio, the present age of each being given.

Let x = the time required, r = the given ratio, m = A's age, and n = B's age.

Then, by the conditions, $$\frac{m+x}{n+x} = r,$$

and $$x = \frac{m - rn}{r - 1}.$$

To discuss this formula:

1st. Suppose that A is now 30 years old and B 20. How long before A will be twice as old as B?

In this case, $r = 2$, $m = 30$, $n = 20$.

$$\therefore \quad x = \frac{30 - 40}{2 - 1} = \frac{-10}{+1} = -10.$$

This means that the event occurred 10 years ago.

2d. Again, suppose A is 30 years old and B 30. How long before A will be 3 times as old as B?

Here $m = 30$, $n = 30$, $r = 3$.

Substituting, $$x = \frac{30 - 90}{3 - 1} = \frac{-60}{+2} = -30.$$

That is, 30 years ago, when the age of each was *zero*, A might be said to be 3 times as old as B. $0 \times 3 = 0$.

3d. Again, let A's age be 30 and B's 30. When will the ratio between their ages be 1?

By the formula, $$x = \frac{30 - 30}{1 - 1} = \frac{0}{0},$$

which, by (Art. 209), is indeterminate, and means that *any time future or past* will satisfy the conditions of the problem, since their ages are *now* equal (or their ratio is 1), and *have been* and *will continue* equal.

4th. Again, let A's age be 30, and B's 15. How long before A will be *twice* as old as B?

Substituting, $$x = \frac{30 - 30}{2 - 1} = \frac{0}{+1} = 0.$$

That is, in zero time, or now, their ages are in that ratio.

5th. Once more, let A's age be 30 and B's age 20. How long before their ages will be equal.

By the formula,

$$x = \frac{30 - 20}{1 - 1} = \frac{+10}{0} = \infty.$$

That is, only at the end of an infinite time; in other words, they will never be of equal age.

PROBLEM OF THE COURIERS.

4. Two couriers, A and B, are travelling to the east, the former m and the latter n miles per hour. At noon, A passes a given point O, and B is a miles in advance of A. How long after noon and how far from O will they be together?

Let t = the required time, and d = the required distance. Then

$$mt - nt = a,$$

and $$t = \frac{a}{m - n}.$$

$$\therefore\ d = \frac{am}{m - n}.$$

To discuss this result, we make the following suppositions:

1st. Let a, m, and n be positive, and $m > n$.

$\therefore$ t and d are positive, and the time of meeting is after noon, and the place east of O. $t > \frac{a}{m}$ and $d > a$.

2d. Let a, m, and n be positive, and $m < n$.

$\therefore$ t and d are negative, and the time is before noon, and the place west of O.

3d. Let $a = 0$, and $m \gtrless n$ (read, "m greater or less than n").

$\therefore$ $t = 0$ and $d = 0$. The time of meeting is noon and the place at O.

4th. Let $m = 0$, and a and n be positive.

$\therefore t = -\frac{a}{n}$ and $d = 0$. The time is before noon and the place as before, but for a different reason. By the former supposition, A *passed* O at noon; by the latter, he remains *at* O all the time.

5th. Let $n = 0$, and a and m be positive.

$\therefore t = \frac{a}{m}$ and $d = a$. B now remains at a miles east of O, but the time is, as it should be, $\frac{a}{m}$ hours after noon.

6th. Let $m = n$, and a be positive.

$\therefore t = \infty$ and $d = \infty$. Their rate of travelling being the same, it will require an infinite time and distance for A to overtake B.

7th. Let $m = n$, and $a = 0$.

$\therefore t = \frac{0}{0}$, and $d = \frac{0}{0}$. Hence,

They are together all the time and everywhere; as they should be, being together at noon and travelling at the same rate.

8th. Let a and m be positive, and n negative.

$\therefore t$ and d are positive, and the result is similar to 1st, but $t < \frac{a}{m}$ and $d < a$.

9th. Let a and n be positive, and m negative.

$\therefore t$ is negative and d positive, and they met before noon, east of O.

10th. Let a be negative and $m = n$.

$\therefore t = -\infty$ and $d = -\infty$.

That is, they started at the same place an infinite time ago, an infinite distance west of O; in other words, they have never been and never will be together in any finite time.

NOTE.—A careful examination of these results will enable the student to discuss the formulas he may obtain by the generalization of problems.

Let the student generalize such of the following problems as are not made general by the statement.

PROBLEMS.

1. A man has 3 times as many half dollars as he has dollars, 5 times as many quarters as he has halves, 3 times as many dimes as quarters, and 5 times as many half dimes as dimes. The whole sum is 44 dollars. How many of each has he?

2. In a family of six persons the average age is 15 years. The mother's age is six years more than the sum of the children's ages, and the father is six years older than the mother. How old is the father?

3. On a certain farm there is twice as much pasturage as tillage, and $33\frac{1}{3}\%$ more woodland than pasturage and tillage together. If 40 acres be taken from the pasturage and added to the tillage, and 50 acres from the woodland and added to the pasturage, the division will be equal. How large is the farm?

4. The sum of two numbers is s, and their difference d. What are the numbers?

5. Divide a into two parts, such that their difference shall equal ma.

6. Divide a into two parts, such that the difference between one part and b shall equal n times the difference between the other part and c.

7. A man has his property invested, $\frac{1}{2}$ in real estate, $\frac{1}{8}$ in government bonds, and the remainder is equally divided between stock in trade and money loaned, on which last investment he realizes, at 6% per annum, \$900 a year. What is the value of his whole estate?

8. A firm doubled its capital during the first year of business; the second year it lost \$100 less than half of the first year's profits; the third year, if the profits had been \$500 more, it would have doubled its capital again; as it was, the capital at the end of the third year was just $2\frac{1}{2}$ times the original investment. What was the capital at first?

9. If, in going a journey, a horse walks half way at the rate of 3 miles an hour, how fast must he go the remaining half to average 4 miles an hour? How fast to average 5 miles an hour? 6 miles?

10. A boy, a years ago, was one-half as old as his mother; now he is one-half as old as his father. How much older is his father than his mother?

11. A merchant sold a bill of goods, one-half at a gain of $33\frac{1}{3}\%$, $\frac{1}{3}$ at 20%, and the rest at 10%; the total gain was \$18.60. What was the amount of the sale?

12. In travelling m miles, the forward wheel of a carriage turns n times more than the hind wheel. If c represent the circumference of the forward wheel, what is the circumference of the hind wheel?

13. A's age is $\frac{1}{a}$ of the sum of B's and C's, and the sum of all their ages is s. What is A's age?

14. A can reap a field in a days, B in b days, C in c days. In what time can they reap it together?

15. Half the contents of a cask containing brandy is drawn off and 20 gallons of water put in; one-half is again drawn off and 50 gallons of water put in, when $\frac{1}{7}$ is brandy? How much was in the cask at first?

16. What number multiplied by a and the product added to b equals c?

17. What number multiplied by m gives a product a less than n times the number?

18. Two men, a miles apart, travel toward each other, one m miles, and the other n miles an hour. In how many hours will they meet?

19. A cask holding 140 gallons is filled with brandy, wine and water. There are 10 gallons more wine than brandy, and as much water as brandy and wine together. What quantity is there of each?

20. A's earnings for the past year are \$100 less than twice B's; B's are \$50 more than one-half C's; and C's are \$25 more than one-third of A's and B's together. What are the earnings of each?

21. What are the two numbers whose sum is 37 and whose difference is 23?

What problem gives a formula for this?

22. A boy had three times as many apples as oranges; he sells 50 apples and 15 oranges, and then has left twice as many apples as oranges. How many of each had he at first?

23. Find a formula for problems like the last.

24. A boy had m times as many apples as oranges. He sold the same number of each, and then had n times as many oranges as apples. Selling again the same number of each as before, he had a apples left. How many did he sell, and how many of each had he at first?

25. A man has four casks. The capacity of the second is $\frac{4}{7}$ of the first, the third $\frac{3}{4}$ of the second and $\frac{9}{16}$ of the fourth, and the first holds 15 quarts more than the third and fourth. How many quarts does each hold?

26. Divide 166 into 5 parts, such that the first shall be 5 less than 3 times the second, 2 less than twice the third, 5 less than 5 times the fourth, and 10 more than three times the fifth.

27. A man rowing uniformly at the rate of four miles an hour, rows down stream one hour, then resting he floats with the current $\frac{1}{2}$ hour. He then rows back in 3 hours. How rapid is the current.

28. A rows 4 and B 3 miles an hour. A is 14 miles farther up stream than B, and they row towards each other till they meet, 4 miles above B's position. How rapid is the current?

29. A besieged garrison had bread to supply each man with 12 oz. per day for 5 weeks, but at the end of one week they lost in a sally 200 men, and the bread remaining was found sufficient to give each man 10 oz. a day for 6 weeks. How many men had they at first?

NOTE.—Several of the above problems may be solved by using one, or more than one unknown quantity. These the student may solve again by using more than one unknown quantity, after completing the subject of Simultaneous Equations.

CHAPTER IX.

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE.

241. In equations containing *two* or *more unknown* quantities, as,

$$x + y = 5,$$
$$x - y = 1,$$

the values of x and y in any one equation are *indeterminate;* for, whatever value be given to one of them, a value can be found for the other which will satisfy the equation.

Thus, in

$$x + y = 5,$$

whatever value be given to y, as a, we have only to make

$$x = 5 - a,$$

and the equation is satisfied.

So in $x - y = 1$, if $x = 1 + a$ and $y = a$, the equation will be satisfied, a being any quantity whatever.

242. But if we assume that both equations are *true at the same time,* that is, for the same values of x and y, the case will be different. The number of values each of the unknown quantities can have will then be *limited,* and they may all be found.

243. When two or more such equations are satisfied *at the same time,* they are called ***Simultaneous.***

It is evident that, if the two equations express conditions obtained from the *same problem,* in which x and y represent the same quantities in both, the values of x and y which will satisfy both equations will also satisfy the conditions of the problem.

244. If the equations actually represent *different conditions,* they will be *independent;* that is, one of them cannot in any way be transformed so as to produce the other.

The equations

$$x + y = 5, \quad \text{and} \quad 2x + 2y = 10,$$

express the *same condition* and are *dependent* equations; but

$$x + y = 5, \quad \text{and} \quad x - y = 1,$$

express *different conditions,* and are *independent.*

245. If a problem requires the finding of several unknown quantities, the solution of the problem will require as many *independent equations* as there are *unknown quantities.*

The problem must therefore furnish a like number of *different conditions,* each of which must be expressed by an equation.

NOTE.—It is not necessary that each of these equations should contain all the unknown quantities. It is only necessary that all the equations should contain them all.

ELIMINATION.

246. The *reduction* of *simultaneous equations* is performed by a process called ***Elimination.***

This process consists in combining two equations, containing two or more unknown quantities, in such manner that one of the unknown quantities shall disappear from the resulting equation.

247. There are four methods of elimination:

1st. By Subtraction.
2d. By Comparison.
3d. By Substitution.
4th. By Division.

CASE I.

248. For ***Elimination by Subtraction*** we have the following

RULE.—*Multiply each of the equations by that quantity which will make the coefficients of the unknown quantity to be eliminated the same in both equations, and subtract one equation from the other.*

1. Given $2x + 3y = 13, \qquad (1)$

$5x - 2y = 4, \qquad (2)$

to find the value of x and y.

BY SUBTRACTION.

(1) × 5 gives	$10x + 15y = 65.$	(3)
(2) × 2 "	$10x - 4y = 8.$	(4)
(3) − (4) "	$19y = 57.$	(5)
(5) ÷ 19 "	$y = 3.$	
Substituting in (1),	$2x + 9 = 13.$	
	$\therefore \quad 2x = 4,$	
	and $\quad x = 2.$	

CASE II.

249. For ***Elimination by Comparison*** we have the following

RULE.—*Find from each equation the value of the unknown quantity to be eliminated in terms of the other unknown and known quantities, and equate these values.*

BY COMPARISON.

Finding x from (1), $\quad x = \dfrac{13 - 3y}{2} \qquad (3)'$

Finding x from (2), $\quad x = \dfrac{4 + 2y}{5}. \qquad (4)'$

Equating $(3)'$ and $(4)'$, $\quad \dfrac{13 - 3y}{2} = \dfrac{4 + 2y}{5}.$

From which, $\quad y = 3,$ as above.

CASE III.

250. For ***Elimination by Substitution*** we have the following

RULE.—*Find from one of the equations the value of the unknown quantity to be eliminated in terms of the other unknown and known quantities, and substitute that value for this unknown quantity in the other equation.*

BY SUBSTITUTION.

Finding x from (1), $x = \frac{13 - 3y}{2}$. (3)′

Substituting in (2), $5\,\frac{13 - 3y}{2} - 2y = 4$. (4)′

Giving as before, $y = 3$.

CASE IV.

251. For ***Elimination by Division*** we have the

RULE.—I. *Clear the equations of fractions and transpose all the terms of each to one member.*

II. *Proceed as if to find the greatest common divisor of the polynomials thus found, and when a remainder is obtained from which one of the unknown quantities has disappeared, put this remainder equal to zero for the equation sought.*

BY DIVISION.

Transposing (1), $2x + 3y - 13 = 0$. (3)‴

Transposing (2), $5x - 2y - 4 = 0$. (4)‴

Dividing (4)‴ × 2 by (3)‴

$$\begin{array}{r|l} 10x - 4y - 8 & \underline{2x + 3y - 13} \\ \underline{10x + 15y - 65} & 5 \\ -19y + 57 & \quad \ldots \text{ Remainder.} \end{array}$$

Putting remainder = 0,

$$-19y + 57 = 0$$
$$19y = 57$$

And, as before, $y = 3$.

NOTES.—1. The reason why this remainder equals *zero* is that the *dividend* and *divisor* each equals *zero;* hence the *remainder* must be zero.

2. The method of elimination to be employed in any case should be that which will render the work most simple.

3. When the method by *Subtraction* is used, the coefficients of the quantity to be eliminated must be made the same in the two equations, not only *in value,* but also *in sign.* If their signs be unlike, they may be made alike by changing all the signs of one of the equations, or the equations may be *added* instead of changing the signs and subtracting.

Reduce the following equations:

2. $5x + 3y = 19,$
$7x - 2y = 8.$

3. $\frac{x}{2} + \frac{y}{3} = 12,$
$\frac{x}{6} - \frac{y}{9} = 0.$

4. $\frac{x}{a} + \frac{y}{b} = m,$
$\frac{x}{c} - \frac{y}{d} = n.$

5. $5x - 7y = -8;$
$5y + 32 = 7x.$

6. $ax + by = h,$
$x + y = d.$

7. $ax + bx = y^2,$
$x = hy.$

8. $\frac{4x}{5} - \frac{2y}{5} = 4,$
$6x = 9y.$

9. $\frac{x + y}{3} = 24,$
$4x + 5y = 116.$

10. $\frac{1}{2}x + 10y = 124,$
$2x + 9y = 124.$

11. $x - 2y = a,$
$2x + 8y = b.$

12. $ax + by - c = 0,$
$a'x + b'y - c' = 0.$

13. $\frac{3}{8}x - \frac{2}{5}y = \frac{1}{10}y,$
$\frac{5}{8}x + \frac{1}{6}y = y.$

14. $y = ax + b,$
$y = a'x + b'.$

15. $y - a = 2(x - b),$
$y - b = 2(x - a).$

16. $\frac{x + 3y}{2} = 7\frac{1}{2},$
$\frac{4x + 5y}{4} = 8.$

17. $\frac{x + 3y}{3} = \frac{c}{3},$
$\frac{x - 3y}{3} = \frac{d}{3}.$

18. $\frac{2x + y}{4} = 4,$
$\frac{3x - 3y}{6} = 1.$

19. $\frac{x + 1}{y} = \frac{1}{3},$
$\frac{x}{y + 1} = \frac{1}{4}.$

PROBLEMS.

1. A number consists of 3 digits. The middle digit is the sum of the other two, the first is twice the last, and inverting the order of the digits gives a number 33 more than half the first number. What is the number?

2. A person spends 50 cents for apples and oranges, buying oranges at 5 cents and apples for 2 cents apiece. If he had taken half as many of each and paid 6 cents apiece for oranges and one cent for apples, they would have cost him 23 cents. How many of each did he buy?

3. What fraction is that whose numerator being increased by 5 and the denominator decreased by 2, equals 1; but whose denominator being increased by 5 and the numerator decreased by 4, becomes $\frac{1}{3}$?

4. Three pipes discharge into the same cistern. The first and second will fill it in $7\frac{1}{2}$ hours, the second and third in 12 hours, and the first and third in $8\frac{12}{21}$ hours. In what time will each pipe fill the cistern?

5. A certain sum of money at interest amounted to $550 in 10 months, and to $560 in 12 months. What was the sum and the rate per cent?

6. Two persons, A and B, can together reap a field of grain in 10 days. They work together 6 days, when A is left to finish the work, which he does in 10 days more. In what time can each reap the field?

7. A and B engage to do a piece of work in 12 days, but after a time, finding themselves unable to accomplish it, C was called in to help them, and the work was finished in time. The rate of working of each was such that A could do the work alone in $\frac{2}{3}$ the time required for B to do it, and C could do it with A in $\frac{7}{8}$ of the time in which he could do it with B, and the three together could do it in 9 days. What part of the work did each one do? How long did C work?

8. A banker has two kinds of money. It takes a pieces of one and b pieces of the other to make a dollar. If c pieces be given for a dollar, how many of each will be used?

9. A, B, and C lunch together. A furnishes 3 loaves and B 2 loaves and a basket of fruit, the whole cost of which was 50 cents; but C, having no provisions, agrees to pay for his share 25 cents in money, when it is found that only 2 cents of this will belong to A. What was the cost of the loaves and what of the fruit?

10. What fraction is that, whose numerator being doubled and denominator increased by 7, the value becomes $\frac{2}{3}$; but the denominator being doubled and the numerator increased by 2, the value becomes $\frac{3}{5}$?

11. A merchant has two kinds of wine. If he mixes a gallons of the first with b gallons of the second the mixture is worth c dollars a gallon; but if he mixes m gallons of the first with n gallons of the second the mixture is worth p dollars a gallon. What is the price of each kind of wine?

12. What two fractions have their sum $1\frac{3}{5}$, and the sum of their numerators equal to the sum of their denominators?

13. A loaned \$500 in two separate sums, the less at 2% more than the other. If the per cent on the greater be increased and that of the less diminished by 1, the whole interest will be increased 25%; but if the per cent on the greater be so increased without changing the other, the interest will be increased $33\frac{1}{3}$%. What were the sums and the rate per cent of each?

14. What is the fraction which becomes $\frac{1}{3}$ when 1 is added to the numerator, and $\frac{1}{4}$ when 1 is added to the denominator?

15. Two men commence business at the same time, A having \$1000 more capital than B. At the end of a year A had lost an amount equal to B's capital and B had gained the same amount, when A's capital is found to be \$2000 more than B's. What was the capital of each?

16. Two men buy a farm in company for \$2000, each putting in all the money he had and giving a mortgage for the balance. If A should pay the mortgage, he would then have invested \$200 more than twice as much as B. If B should pay the mortgage, he would have invested \$200 more than A. How much cash did each put in, and how much was the mortgage?

THREE OR MORE UNKNOWN QUANTITIES.

252. When there are three or more unknown quantities, and a like number of equations, the reduction is made by the following

RULE.—I. *Eliminate the same unknown quantity from different pairs of the given equations, thus forming a set of equations independent of this unknown quantity, and one less in number than the given equations.*

II. *From these, in like manner, eliminate another unknown quantity, and so continue till an equation is found with but one unknown quantity.*

III. *From this find the value of that unknown quantity, and substitute it in a previous equation to find the value of another unknown quantity. Substitute these two to find a third, and so on till all are found.*

The following example will illustrate this process:

1. Given

$$x + 2y - 3w + z = 4 \quad (1)$$
$$2x - y + 2w - 3z = 1 \quad (2)$$
$$5x - 3y - w - 2z = 11 \quad (3)$$
$$3x + 4y - 5w + 6z = -9 \quad (4)$$

to find the value of x, y, w, and z.

(1) × 3,	$3x + 6y - 9w + 3z = 12$	(5)
(2) + (5),	$5x + 5y - 7w = 13$	(6)
(1) × 2,	$2x + 4y - 6w + 2z = 8$	(7)
(3) + (7),	$7x + y - 7w = 19$	(8)
(1) × 6,	$6x + 12y - 18w + 6z = 24$	(9)
(9) − (4),	$3x + 8y - 13w = 33$	(10)
(8) − (6),	$2x - 4y = 6$	(11)
(6) × 13,	$65x + 65y - 91w = 169$	(12)
(10) × 7,	$21x + 56y - 91w = 231$	(13)
(12) − (13),	$44x + 9y = -62$	(14)
(11) × 22,	$44x - 88y = 132$	(15)
(14) − (15),	$97y = -194$	(16)
(16) ÷ 97,	$y = -2$	

Substituting in (11), $2x + 8 = 6$
And $x = -1$
Substituting in (8), $-7 - 2 - 7w = 19$
And $w = -4$
Subst. in (1), $-1 - 4 + 12 + z = 4$
And $z = -3$

$\therefore\ x = -1,\ y = -2,\ w = -4,$ and $z = -3,$ *Ans.*

2. $2x + 3y - 4z = 8,$
$3x - 4y + 2z = 3,$
$4x - 2y - 3z = 5.$

3. $ax + by + cz = m,$
$bx + cy + az = n,$
$cx + ay + bz = r.$

4. $x + y = 12,$
$y - z = 3,$
$z + u = 7,$
$u + x = 8.$

5. $\frac{x}{2} - \frac{y}{3} + \frac{z}{4} = 1,$
$\frac{x}{3} - \frac{y}{4} + \frac{z}{2} = \frac{23}{12},$
$\frac{x}{4} + \frac{y}{2} - \frac{z}{3} = \frac{2}{3}.$

6. $5x - 7y + z + u = 2,$
$7x - 3y - 3z + 2u = 2,$
$4x - 2y = 2,$
$x + 5y - 2z = 2.$

7. $\frac{1}{x} + \frac{1}{y} = \frac{1}{a},$
$\frac{1}{y} + \frac{1}{z} = \frac{1}{b},$
$\frac{1}{z} + \frac{1}{x} = \frac{1}{c}.$

8. $ax + by + cz = 0,$
$a'x + b'y + c'z = 0,$
$a''x + b''y + c''z = 0.$

9. $ax + bx - cy = m,$
$ay + by - cx = n.$

10. $xyz = a(xy + yz - xz) = d(xy + xz - yz)$
$= c(xz + yz - xy).$

PROBLEMS.

1. A number consists of 4 digits. The first is half the second; the third, twice the second plus the first; the fourth, the sum of the second and third; and the sum of the digits is 15. What is the number?

2. Find three numbers, such that $\frac{1}{2}$ the sum of the first and second shall be 50, $\frac{1}{2}$ of the second and third shall be 65, and $\frac{1}{2}$ of the first and third shall be 55.

3. The average age of A, B, and C is a. The average age of A and B is b, and of B and C is c. What are their ages?

4. Divide the number 150 into three parts, such that $\frac{1}{2}$ the first shall be $\frac{1}{3}$ of the second, and $\frac{1}{6}$ of the second shall be $\frac{1}{5}$ of the third.

5. A person has 2 horses and 2 saddles, all of which are worth \$265. The poorer horse and better saddle are worth \$5 less than the better horse and poorer saddle, while the better horse and better saddle are worth \$45 more than the poorer horse and poorer saddle, and the horses are worth 5 times as much as the saddles. What is the value of each horse and saddle?

6. Three brothers, A, B, and C, bought a farm for \$1000, A's money with $\frac{1}{9}$ of B's and $\frac{1}{6}$ of C's would pay for it; so also would B's money with $\frac{1}{10}$ of C's and $\frac{1}{20}$ of A's, or C's with $\frac{1}{5}$ of B's and $\frac{1}{8}$ of A's. How much money had each?

7. A, B, and C bought a farm for a dollars. A's money with $\frac{1}{m}$ of B's and $\frac{1}{n}$ of C's, or B's money with $\frac{1}{m'}$ of A's and $\frac{1}{n'}$ of C's, or C's money with $\frac{1}{m''}$ of A's and $\frac{1}{n''}$ of B's will pay for the farm. How much money had each?

8. A and B can do a certain piece of work in a days; B and C in b days; C and D in c days; and A and C in d days. In what time can each do the work alone, and how long would they be in doing it, working all together?

9. Three men began trade at the same time. A had \$2000 more than twice as much capital as B, and C had \$500 less than A and B together. The first year A gained as much as B's original capital, B gained as much as A's capital, and C gained as much as A's and B's capital together, when each had an equal sum. How much had each at first?

10. A doctor visits a patient, and when half way home he is overtaken by a messenger, and called to return $3\frac{1}{2}$ miles to visit a second patient; and again, when half way home, he is called to return to visit a third patient, 3 miles farther away than the first. On his return to his office, he finds he has driven 20 miles. How far did each patient live from his office?

CHAPTER X.

POWERS AND ROOTS.

253. A ***Power*** is the *product* of any number of the *equal factors of a quantity.*

254. ***Powers*** are expressed by *exponents,* which show *what equal factors* are taken. Hence,

A quantity with any exponent is a power.

255. The *quantity itself* is called the ***Base*** of the power.

256. The word *power* is used with reference to the *effect* of a quantity as a ***Factor,*** while its *effect* as a ***Term*** is called its *value* or *magnitude.*

Thus, if $\frac{2}{3}$ of the *factors* of a are required, it is written $a^{\frac{2}{3}}$; if two-thirds of a as *a term,* it is written $\frac{2}{3}a$.

257. The operation of finding any part of the factors of a quantity is similar to that of finding any part of its terms.

For example, $\frac{2}{3}$ *of the terms of* 27 are found by separating 27 into *three equal terms* ($27 \div 3 = 9$), and combining *two* of them ($9 + 9 = 18$, or $9 \times 2 = 18$).

So $\frac{2}{3}$ *of the factors* of 27 are found by separating 27 into *three equal factors* ($27^{\frac{1}{3}} = 3$), and combining *two* of them ($3 \times 3 = 9$).

258. ***Evolution*** is the process by which a quantity is *separated into* any number of *equal factors.*

259. ***Involution*** is the process of finding the *product of equal factors.*

NOTES.—1. If the *exponent* of the required power have 1 for its *denominator* (that is, if the *exponent* be *integral*), the factor to be found by *evolution* will be *the base*, and the work of finding the required power will be *wholly involution*.

2. If the exponent have 1 for a *numerator*, the work will be wholly *evolution*.

3. If both *numerator* and *denominator* be 1 (that is, if the exponent be unity), the power is already found in the quantity itself; that is,

260. The ***First Power*** is the quantity itself, or *base*.

261. As a *coefficient* shows by its *denominator* the *number* of ***equal terms*** into which a quantity is to be separated, and by its *numerator* the *number* of these terms that are to be taken, so an *exponent* shows by its *denominator* the *number* of ***equal factors*** into which a quantity is to be separated, and by its *numerator how many* of these *factors* are to be taken.

262. A ***Root*** is one of the *equal factors* of a quantity; as, $a^{\frac{1}{2}}$, $a^{\frac{1}{3}}$, $a^{\frac{1}{4}}$, etc., which may be read, "*a one-half power, a one-third power*," etc.; or, "*the square (or second) root of a, the cube (or third) root of a, the fourth root of a*," etc. (Art. 58, 59.)

263. Quantities with fractional exponents are called ***Radical Quantities.***

They may be expressed by the *radical sign*, the *numerator* of the exponent remaining as an exponent of the base, and the *denominator* being placed over the sign, except when it is two (2), in which case it is omitted.

Thus, $a^{\frac{1}{2}} = \sqrt{a}$; $a^{\frac{2}{3}} = \sqrt[3]{a^2}$; $a^{\frac{3}{2}} = \sqrt{a^3}$, etc.

264. A *Power* which can be expressed without a fractional exponent, as $a^{\frac{4}{2}}$, is called ***Rational,*** and may be written a^2.

When it cannot be so expressed, it is called ***Irrational,*** or ***Surd;*** as, $a^{\frac{1}{3}}$, $a^{\frac{2}{3}}$, etc.

Thus, $8^{\frac{1}{3}}$, $4^{\frac{1}{2}}$, and $8^{\frac{2}{3}}$ are *rational;* but $8^{\frac{1}{2}}$, $4^{\frac{1}{3}}$, and $9^{\frac{2}{3}}$ are *irrational*.

265. To make the use of exponents familiar to the student, let the teacher ask such questions as the following:

1. What is the value of $16^{\frac{1}{2}}$? $16^{\frac{1}{4}}$? $16^{\frac{3}{4}}$?

2. What is the value of $16^{\frac{2}{4}}$? $16^{\frac{3}{2}}$? 16^{0}?

3. What is the value of 16^{-1}? 16^{-2}? 16^{2}? $16^{-\frac{1}{2}}$? $16^{-\frac{3}{4}}$? 16^{-0}?

4. What is the value of $(16^{-1})^{2}$? $(16^{-1})^{-1}$? $(16^{-1})^{-2}$? $32^{\frac{3}{5}}$? $32^{-\frac{3}{5}}$? etc., etc.

POWERS OF MONOMIALS.

266. ***Any power*** of a quantity may be *expressed* by giving it the *exponent* of the required power; as,

$$(8a^3b^2)^{\frac{2}{3}}, \qquad (a^2b + 2b^2)^{\frac{2}{3}}.$$

In the case of the monomial, the parenthesis may be removed by applying the exponent to each factor separately.

Thus, $(8a^3b^2)^{\frac{2}{3}} = 8^{\frac{2}{3}}(a^3)^{\frac{2}{3}}(b^2)^{\frac{2}{3}} = 4a^2b^{\frac{4}{3}}$. (Arts. 117, 118.) Hence,

To find any power of a monomial we have the following

RULE.—*Multiply the exponent of each factor of the monomial by the exponent of the required power.*

NOTES.—1. It will be observed that the numerical factor or coefficient must be included, and that when this becomes rational the required power may be taken, as in the example above.

2. This rule applies to all powers of a monomial, integral or fractional, and includes the work of *evolution* and *involution*.

5. Find the m^{th} power of $(-a^3b^2)$.

6. Find the third power of $(x - y)^{\frac{2}{3}}$.

267. When the quantity whose power is taken is regarded as having a *directive sign*, the same rule applies to the sign.

Thus, $(-a^3b^2)^{\frac{2}{3}}$, which equals $-^{\frac{2}{3}}a^2b^{\frac{4}{3}}$.

We have already seen the force of the signs with integral exponents. Their meaning with fractional exponents will be considered hereafter.

POWERS OF BINOMIALS.

268. ***Any power*** of a ***binomial*** may be found by the ***Binomial Formula,*** given below, in which a and x represent the *terms* of the binomial and n the *exponent* of the power, which may be *integral* or *fractional, positive* or *negative.*

Note.—We are not yet prepared to demonstrate this formula, but the student may verify it for any *integral* powers of $(a+x)$ by actual multiplication, and by committing it to memory it can be used with equal facility before or after demonstration.

BINOMIAL FORMULA.*

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1\cdot 2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}x^3, \text{ etc.}$$

269. An inspection of the formula will show that,

I. *The exponents of the leading letter* (a), *beginning with the exponent of the power* (n), *decrease by unity in the successive terms.*

II. *The exponents of the following letter* (x), *beginning with* 0, *increase in like manner by unity.*

III. *The sum of the exponents in any term equals the exponent of the required power.*

IV. *The coefficient of any term (after the first, whose coefficient is* 1) *is the product of the coefficient of the preceding term by the exponent of the leading letter in that term, divided by the exponent of the following letter in the term itself.*

V. *The number of terms in the series will be* $n+1$ *when the exponent of the power is integral and positive, and infinite in all other cases.*

VI. *The* m^{th} *or general term is*

$$\frac{n(n-1)(n-2)\ldots\ldots(n-m+2)}{1\cdot 2\cdot 3\cdot 4\ldots\ldots(m-1)}a^{n-m+1}x^{m-1}.$$

Note.—This formula translated into common language is called the Binomial Theorem.

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* This formula was discovered by Sir Isaac Newton. (See p. 305, Note 1.)

270. Although the terms of the formula are all $+$, it must be remembered that as a, x, and n may one or more of them be negative, the sign of each term must be determined by the general *Rule of Signs.* (Art. 91.)

271. SIGNS.—I: *If both terms of the binomial are positive, and the exponent integral and positive, the terms of the series will all be positive.*

II. *If both terms of the binomial are positive and the exponent negative (whether integral or fractional), the terms of the series will be alternately positive and negative.*

III. *If both terms of the binomial are positive, and the exponent is a positive fraction, the terms of the series will be positive until the term whose number is the next integer greater than* $(n + 2)$. *This term will be negative, and the following terms will be alternately positive and negative.*

IV. *If the second term of the binomial be negative (the first being positive), the alternate terms, beginning with the second, will have signs opposite those given in the cases above.*

NOTE.—Let the student verify each of the statements in Arts. 269 and 271 by an examination of the *formula.*

INVOLUTION OF POLYNOMIALS.

272. In finding the powers of polynomials, we must perform the involution and evolution separately. The square of a polynomial may be written by Theorem IV, Art. 156.

273. The cube of a polynomial may be written by the following

RULE.—*The cube of a polynomial is equal to the sum of the cubes of its several terms, plus three times the products of the square of each term by each of the other terms, plus six times the products of the terms taken three at a time.*

NOTE.—Higher integral powers of polynomials can be found by actual multiplication.

274. A ***Multinomial Theorem,*** by which any power of a polynomial may be written in the same manner as powers of a binomial by the Binomial Formula, is sometimes given, but it is of little use in ordinary mathematical operations.

EXAMPLES.

Develop the following by the Binomial Formula:

1. $(a + b)^5 = a^5 + 5a^4b + \dfrac{5\cdot 4a^3b^2}{\underline{|2}} + \dfrac{5\cdot 4\cdot 3a^2b^3}{\underline{|3}}$

$$+ \frac{5\cdot 4\cdot 3\cdot 2ab^4}{\underline{|4}} + \frac{5\cdot 4\cdot 3\cdot 2\cdot 1b^5}{\underline{|5}}$$

$$= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5, \textit{ Ans.}$$

2. $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$, *Ans.*

3. $(a + b)^{\frac{1}{2}} = a^{\frac{1}{2}} + \frac{1}{2}a^{-\frac{1}{2}}b + \dfrac{\frac{1}{2}(\frac{1}{2} - 1)}{\underline{|2}}a^{-\frac{3}{2}}b^2 + \text{etc.}$

$$= a^{\frac{1}{2}} + \frac{b}{2a^{\frac{1}{2}}} - \frac{b^2}{8a^{\frac{3}{2}}} + \text{etc.}, \textit{ Ans.}$$

4. $(a + b - c + d)^2 = a^2 + 2ab - 2ac + 2ad + b^2 - 2bc$
$+ 2bd + c^2 - 2cd + d^2$, *Ans.*

5. $(a + x)^2$ and $(a - x)^2$.
6. $(2a + b)^2$ and $(a - 2b)^2$.
7. $(a + b)^{\frac{1}{3}}$ and $(a - b)^{\frac{1}{3}}$.
8. $(x + y)^6$ and $(x - y)^7$.
9. $(2a + 2b)^5$ and $(3a - 3b)^8$.
10. $(a + x)^{-1}$ and $(a - x)^{-2}$.
11. $(a + x)^{\frac{2}{3}}$ and $(a - x)^{\frac{2}{3}}$.
12. $(a + c)^{\frac{5}{3}}$ and $(a + c)^{-\frac{5}{3}}$.
13. $(a - c)^{-\frac{5}{3}}$ and $(a - c)^{\frac{5}{3}}$.
14. $(ax + by)^{\frac{1}{3}}$ and $(ax - by)^{-\frac{1}{3}}$.

Expand the following by Theorem IV and Rule, Art. 156:

15. $(a + 2b - 4c)^2$.
16. $(2a - 3ax + b^2 - xy + z)^2$.
17. $(ax + by - 3z + 5)^2$.
18. $(a + b - c + d)^3$.
19. $(2x - 3y + z)^3$.
20. $(ax + by + z + m - n)^3$.

EVOLUTION OF POLYNOMIALS.

275. To find any ***Root*** of a ***Polynomial***, the *monomial* factors should be removed and their roots taken as factors of the required root.

Let P represent any polynomial whose n^{th} root is required, and from which the monomial factors have been removed, and let x represent one term and y the algebraic sum of the remaining terms of the root. Then

$$P = (x + y)^n = x^n + nx^{n-1}y + \text{etc.},$$

in which y may be a polynomial. (Art. 268.)

Now if none of the literal factors of x are found in any term of y, the root sought may be found at once by taking the n^{th} root of that term of P represented by x^n for one term of the root, and dividing all the terms represented by $nx^{n-1}y$ by nx^{n-1} for the rest of the root.

But, as some of the factors of x may be found in one or more terms of y, $(x+y)^n$ may have similar terms, which by uniting will prevent finding by inspection the value of $nx^{n-1}y$. And yet, since there will be at least one term of y which does not contain x (the monomial factors having been removed), we shall always be able to find at least *two* terms of the root, which being raised to the n^{th} power and subtracted from P, will give a remainder from which other terms of $nx^{n-1}y$ can be found. A repetition of this process will give the whole root.

276. This process may be expressed by the following

RULE.—I. *Find by inspection those terms of P which are the n^{th} powers of the terms of the root, and take the roots of these for terms of the root.*

II. *Representing these terms by x, x_1, x_2, etc., find the terms representing $nx^{n-1}y$, $nx_1^{n-1}y$, etc., and divide them respectively by nx^{n-1}, nx_1^{n-1}, etc., for other terms of the root.*

III. *Raise the part of the root so found to the n^{th} power, subtract it from P, and find in the remainder other terms of the form $nx^{n-1}y$, and divide as before.*

NOTES.—1. The terms to be found by Part I of this rule will be those containing the highest powers of the different literal factors; or, if there are two terms that contain higher powers of any letter than any others, the one containing the least number of other literal factors will be the term required.

2. The terms to be found in Part II will be those containing the next lower powers of the letters.

277. By making $n = 2$, we have the rule for taking the *square root*, and $n = 3$ gives the rule for the *cube root*.

278. ***Any Root*** whose index is the *product* of *two or more factors* may be found by taking successively the roots indicated by those factors.

Thus, the sixth root is the cube root of the square root.

EXAMPLES.

1. Find the cube root of

$$8a^6 - 36a^5b + 33a^2b^4 + 66a^4b^2 - 63a^3b^3 - 9ab^5 + b^6.$$

SOLUTION.—By Part I of rule we find that $8a^6$ and b^6 must be cubes of terms of the root, and we get from them $2a^2 + b^2$ as a part of the root.

By Part II we take the term containing the next lower power of a, and divide it by $n(2a^2)^{n-1} = 3(2a^2)^2 = 12a^4$ (n in this case being 3). This gives $-36a^5b \div 12a^4 = -3ab$.

Also, taking the term containing the next lower power of b and dividing, we have $-9ab^5 \div 3b^4 = -3ab$.

By each of these last two steps we find $-3ab$ as another term of the root. This does not mean that $-3ab$ is found twice in the root, for the same term of the root will frequently be found more than once by the method above.

By cubing the root, $2a^2 + b^2 - 3ab$, we shall find the given polynomial.

2. Find the cube root of $8x^{\frac{3}{2}}y - 36x^{\frac{4}{3}}y^{\frac{2}{3}} + 12xy^{\frac{7}{6}} - 27x + 54x^{\frac{7}{6}}y^{\frac{1}{3}} + 27x^{\frac{2}{3}}y^{\frac{1}{2}} + y^{\frac{3}{2}} + 6x^{\frac{1}{2}}y^{\frac{4}{3}} - 9x^{\frac{1}{3}}y - 36x^{\frac{5}{6}}y^{\frac{5}{6}}$.

SOLUTION.—From Part I of rule we have $2x^{\frac{1}{2}}y^{\frac{1}{3}} - 3x^{\frac{1}{3}} + y^{\frac{1}{2}}$ as terms of the root, and Part II gives no other terms. This root cubed gives the polynomial.

3. Find the square root of $a^2 - 2a^2x - 2a^3x + 4a^3x^2 - 2ac + a^2x^2 - 2a^3x^3 + 2acx + a^4x^2 - 2a^4x^3 + 2a^2cx + a^4x^4 - 2a^2cx^2 + c^2$.

SOLUTION.—By Part I of the rule, $\pm c$ is one term of the root.

By Part II, dividing the terms containing c first power (viz., $-2ac + 2acx + 2a^2cx - 2a^2cx^2$) by $\pm 2c$ gives the other terms, $\mp a \pm ax \pm a^2x \mp a^2x^2$.

$$\therefore \quad \pm(c - a + ax + a^2x - a^2x^2) = \text{the root required.}$$

EXAMPLES.

Find the square roots of

1. $a + 2a^{\frac{1}{2}}x^{\frac{1}{2}} + x$.
2. $a - 2a^{\frac{1}{2}}x^{\frac{1}{2}} + x$.
3. $a^{2n} \pm 2a^nx^n + x^{2n}$.
4. $a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4$.
5. $a^6 + 6a^5x + 15a^4x^2 + 20a^3x^3 + 15a^2x^4 + 6ax^5 + x^6$.

Find also the cube root of the last polynomial and of the following :

6. $a^6 + 3a^4x^2 + 3a^2x^4 + x^6$.
7. $a^{\frac{3}{2}} + \frac{3}{2}a + \frac{3}{4}a^{\frac{1}{2}} + \frac{1}{8}$.
8. $a^3 - 3a^2x + 3ax^2 - x^3$.

Find the fifth root of

9. $a^5 - 5a^4 + 10a^3 - 10a^2 + 5a - 1$.

279. The *square root* of numerical binomials of the form $a \pm m\sqrt{b}$ may often be found by *separating the rational term into parts, to give it the form* $x^2 \pm 2xy + y^2$.

10. Find the square root of $7 + 4\sqrt{3}$.

SOLUTION. $7 + 4\sqrt{3} = 4 + 4\sqrt{3} + 3 = (2 + \sqrt{3})^2$.

$\therefore \quad (7 + 4\sqrt{3})^{\frac{1}{2}} = 2 + \sqrt{3}$, *Ans.*

11. Find the square root of $11 - 6\sqrt{2}$.
12. Find the square root of $41 \pm 12\sqrt{5}$.
13. Find the square root of $33 \pm 20\sqrt{2}$.

SIGNS OF ROOTS.

280. A ***Power*** of a quantity, by definition, is that quantity affected by any *exponent* whatever; while a ***Root*** is the quantity affected by a fractional exponent whose numerator is 1.

NOTE.—We use the word *root* in the present discussion to distinguish such powers, and not, as it is frequently used, for fractional powers in general.

281. In considering *what sign* shall be given to fractional powers, we first find what are the proper signs of roots; and as all fractional powers are *integral* powers of roots, we may find their signs by Art. 91.

For example, $a^{\frac{3}{5}}$ is the third power of the fifth root of a. When we know the sign of the fifth root of a, the third power of that sign, or that sign taken three times, will be the sign of $a^{\frac{3}{5}}$.

282. The sign of a root is found by the general rule, giving the sign of the quantity whose root is taken the exponent of the root. (Art. 90.) Thus,

$$(-a^3)^{\frac{1}{3}} = -^{\frac{1}{3}} a; \qquad (+a^2)^{\frac{1}{3}} = +^{\frac{1}{3}} a, \quad \text{etc.}$$

If we have a positive quantity (a), it may be written

$$+a, \qquad -^2 a, \qquad -^4 a, \qquad -^6 a, \quad \text{etc.};$$

or, for uniformity, we may write,

$$-^0 a,^* \qquad -^2 a, \qquad -^4 a, \qquad -^6 a, \quad \text{etc.}; \quad (1)$$

using any even power of $-$ to express the positive direction.

So, also, if a be negative, it may be written

$$-a, \qquad -^3 a, \qquad -^5 a, \qquad -^7 a, \quad \text{etc.} \qquad (2)$$

283. If now we take any root of $+a$, we may use for $+a$ any of the expressions in (1); or if we take a root of $-a$, we may use any of the expressions in (2). This will give for the n^{th} root of $+a$,

$$-^{\frac{0}{n}} a^{\frac{1}{n}}, \qquad -^{\frac{2}{n}} a^{\frac{1}{n}}, \qquad -^{\frac{4}{n}} a^{\frac{1}{n}}, \qquad -^{\frac{6}{n}} a^{\frac{1}{n}}, \quad \text{etc.}; \qquad (3)$$

and for the n^{th} root of $-a$,

$$-^{\frac{1}{n}} a^{\frac{1}{n}}, \qquad -^{\frac{3}{n}} a^{\frac{1}{n}}, \qquad -^{\frac{5}{n}} a^{\frac{1}{n}}, \qquad -^{\frac{7}{n}} a^{\frac{1}{n}}, \quad \text{etc.}, \qquad (4)$$

n being any integral number.

---

* As a^0, which has no power as a factor, equals 1, so $+$, which has no power as a factor of direction, may be represented by $-^0$.

284. From this we see that we may have different signs for the same root, depending on the sign we use to express the direction of the quantity whose root is taken. These are called the *different roots* of a quantity, the only difference however being in the sign.

285. To find the number of these roots, examine first the signs of the roots of a positive quantity. The first sign in (3) is

$$-^{\frac{0}{n}} = -^{0} = +. \quad \text{Hence,}$$

One root, of whatever degree, of a positive quantity is positive.

286. The other exponents of the signs in (3) will be fractions until we come to one whose numerator is divisible by n. If n be an even number, we shall have the numerator n, giving a root whose sign is

$$-^{\frac{n}{n}} = -^{1} = -. \quad \text{Hence,}$$

One of the even roots of a positive quantity is negative.

If n be odd, the second, and if even, the third integral exponent of $-$ will be $\frac{2n}{n} = 2$, and we shall have the root whose sign is $-^{2} = +$, which is the same as the first root. The exponents of the sign after $\frac{2n}{n}$ or 2 will be, for all values of n,

$$2 + \frac{2}{n}, \qquad 2 + \frac{4}{n}, \qquad \text{etc.}$$

287. As it will not affect the signs to drop from the exponent an even number, they may be written

$$\frac{2}{n}, \qquad \frac{4}{n}, \qquad \text{etc.,}$$

showing that the series will repeat itself from that point, and the number of different roots, or more properly of *different signs* for the root, will be n.

288. If we now examine the signs of the roots of a negative quantity as given in (4), we shall find the first integral exponent, when n is an odd number, to be

$$\frac{n}{n} = 1,$$

giving the sign $\quad -^1 = -. \quad$ Hence,

One of the odd roots of a negative quantity is negative.

289. When n is an even number, we shall find in (4) no integral exponents of the sign; but whatever be the value of n, we shall find the exponent

$$\frac{2n+1}{n} = 2 + \frac{1}{n},$$

after which we shall get

$$2 + \frac{3}{n}, \qquad 2 + \frac{5}{n}, \qquad 2 + \frac{7}{n}, \qquad \text{etc.};$$

and as we may omit the 2, the exponents repeat themselves, and we have as before the number of roots equal to n. Hence, in general,

290. *Every quantity has as many roots as there are units in the degree of the root.*

This may be illustrated as follows:

1. $\quad (+4)^{\frac{1}{2}} = (-^2 4)^{\frac{1}{2}} = (-^4 4)^{\frac{1}{2}} = (-^6 4)^{\frac{1}{2}}$, etc.,

which may be written

$$+2, \qquad -2, \qquad -^2 2, \qquad -^3 2, \qquad \text{etc.},$$

the last two being the same as the first two, and we have only the two roots $+2$ and -2.

2. $\quad (-4)^{\frac{1}{2}} = (-^3 4)^{\frac{1}{2}} = (-^5 4)^{\frac{1}{2}} = (-^7 4)^{\frac{1}{2}}$, etc.; or

$$-^{\frac{1}{2}} 2, \qquad -^{\frac{3}{2}} 2, \qquad -^{\frac{5}{2}} 2, \qquad -^{\frac{7}{2}} 2, \qquad \text{etc.};$$

which give only $-^{\frac{1}{2}} 2$ and $-^{\frac{3}{2}} 2$, the others being the same as these.

3. $(+8)^{\frac{1}{3}} = (-^{2}8)^{\frac{1}{3}} = (-^{4}8)^{\frac{1}{3}} = (-^{6}8)^{\frac{1}{3}}$, which become

$$+2, \qquad -^{\frac{2}{3}}2, \qquad -^{\frac{4}{3}}2, \qquad -^{2}2,$$

and we get three cube roots of $+8$.

4. $(-8)^{\frac{1}{3}} = (-^{3}8)^{\frac{1}{3}} = (-^{5}8)^{\frac{1}{3}} = (-^{7}8)^{\frac{1}{3}}$; or,

$$-^{\frac{1}{3}}2, \qquad -2, \qquad -^{\frac{5}{3}}2, \qquad -^{\frac{7}{3}}2 = (-^{\frac{1}{3}}2),$$

and we have also three cube roots of -8.

In the same way the student may find 4 fourth roots of 16, and 5 fifth roots of 32, etc.

291. We have taken rational quantities for illustration, but the same would of course be true for irrational quantities.

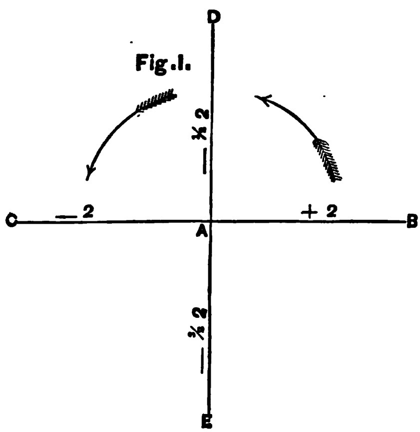
We have found that the signs of some of these roots are $+$ and some $-$, while most of them can only be expressed by the sign $-$ with a fractional exponent.

IMAGINARY QUANTITIES.

292. The force of the sign $-$, with any integral exponent, has already been explained. (Arts. 90, 91.) It now remains to consider what interpretation shall be given it when affected by a fractional exponent.

Let AB, Fig. 1, be a line whose length is 2, and whose direction is *positive*, reckoned from A to B. Then AC, lying in the opposite direction, will $= -2$, and these lines will represent the sq. roots of $+4$. Since the sign $-$, in reversing the line AB, turns it in the direction indicated by the arrow (Art. 90), when it has expended $\frac{1}{2}$ its power, the line will have the direction AD, which will therefore be expressed by $-^{\frac{1}{2}}2$. In like manner, the $\frac{3}{2}$ power of

Fig. I.



this sign ($-\frac{3}{2}$) brings the line to AE. These lines represent the square roots of -4.

In like manner, all fractional powers of $-$ indicate directions out of the line $+$ and $-$, which are found by taking *such part* of a reversal as the exponent of the power indicates.

293. The directions of the *cube roots* of a *positive quantity* are represented by the lines AB, AD, and AE, Fig. II.

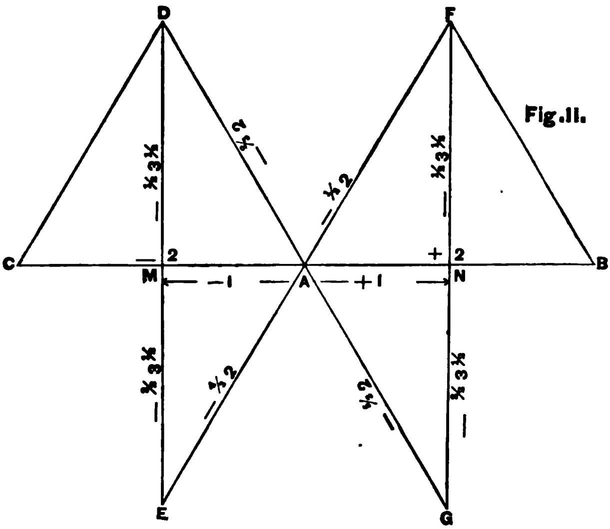


Fig. II.

If these lines be taken 2 units in length, they will represent the cube roots of $+8$ both in magnitude and direction.

So also AF, AC, and AG represent the cube roots of -8.

294. A ***Real Quantity*** is one whose sign is either $+$ or $-$.

295. An ***Imaginary Quantity*** is one whose *sign* is $-$, with some *fractional exponent.*

NOTE.—In the solution of problems, quantities are always considered as lying in a certain line in one of two opposite directions, called *positive* and *negative.* (Art. 86, 3°.) Any quantity not in that line is regarded as *unreal,* and is therefore called *imaginary.* Such a quantity in a result indicates the introduction of some *impossible* condition.

296. From (Arts. 281–289) we deduce the following

PRINCIPLES.

1°. *Of the even roots of a positive quantity, two are real, one + and the other —.*

2°. *Of the odd roots of a positive quantity, one only is real, and its sign is +.*

3°. *Of the odd roots of a negative quantity, one only is real and its sign is —.*

4°. *All the even roots of a negative quantity are imaginary.*

297. In mathematical computations, the *real roots* of quantities are used in all cases, when there are such; hence the only case which necessitates the use of *imaginary* quantities is that which requires the *even root* of a *negative quantity.*

For this reason, an imaginary quantity is often defined as the *even root* of a *negative* quantity.

298. In Fig. II, if the lines DE and FG be drawn perpendicular to BC, then

$$\mathrm{AN} = +1; \qquad \mathrm{MD} = (-3)^{\frac{1}{2}};$$
$$\mathrm{AM} = -1; \qquad \mathrm{NG} = -(-3)^{\frac{1}{2}};$$
$$\mathrm{NF} = -^{\frac{1}{2}}3^{\frac{1}{2}} = (-3)^{\frac{1}{2}}; \qquad \mathrm{ME} = -(-3)^{\frac{1}{2}}.$$

NOTE.—The student who understands Trigonometry may verify these values.

$$\mathrm{AN} + \mathrm{NF} = 1 + \sqrt{-3};$$
$$\mathrm{AN} + \mathrm{NG} = 1 - \sqrt{-3};$$
$$\mathrm{AM} + \mathrm{MD} = -1 + \sqrt{-3};$$
$$\mathrm{AM} + \mathrm{ME} = -1 - \sqrt{-3}.$$

For all purposes of mathematical calculation,

$$\mathrm{AN} + \mathrm{NF} = \mathrm{AF};$$
$$\mathrm{AM} + \mathrm{MD} = \mathrm{AD};$$
$$\mathrm{AN} + \mathrm{NG} = \mathrm{AG};$$
$$\mathrm{AM} + \mathrm{ME} = \mathrm{AE},$$

299. Hence we have the following equations:

$$-^{\frac{1}{3}}2 = 1 + \sqrt{-3}; \qquad (1)$$
$$-^{\frac{5}{3}}2 = 1 - \sqrt{-3}; \qquad (2)$$
$$-^{\frac{2}{3}}2 = -1 + \sqrt{-3}; \qquad (3)$$
$$-^{\frac{4}{3}}2 = -1 - \sqrt{-3}. \qquad (4)$$

The first two are the imaginary cube roots of -8 and the last two of $+8$.

NOTES.—1. It is evident that travelling over the distance AF from A is equivalent to travelling over AN and NF successively, so far as the *result* is concerned.

2. But it is not so evident that the *second members* of these equations may be used for the *first members* in mathematical computations. The student may, however, verify the statement. For example,

1. Add the 1st and 4th, and we have,			$0 = 0$.
2. Add the 2d and 3d, and	"	"	$0 = 0$.
3. Cube the 1st or 2d, and	"	"	$-8 = -8$.
4. Cube the 3d or 4th, and	"	"	$8 = 8$.

Observe in adding that $-^{\frac{1}{3}}2$ and $-^{\frac{4}{3}}2$ lie in opposite directions, and being equal in distance, their sum is zero.

300. The forms in the second members of these equations are always used instead of those in the first members. The advantage of this will be readily seen in the fact that the second form has its *imaginary* part always a *square root,* which will lie along a line at right angles to BC, in one of two opposite directions.

Such quantities may therefore be added and subtracted like real quantities.

Thus, we may add

$$\begin{array}{ll} & -1 + \sqrt{-3} \\ \text{and} & \underline{-1 - \sqrt{-3}} \\ \text{Sum,} & -2 \end{array}$$

which is represented by the line AC. But we cannot by any algebraic process reduce $-^{\frac{2}{3}}2$ and $-^{\frac{4}{3}}2$ to one term.

NOTES.—1. We may, however, see that by a different method of addition (not algebraic), the sum of $-^{\frac{2}{3}} 2$ and $-^{\frac{4}{3}} 2$ is -2; for if we go from A to D, and then from D a distance and direction equal to $-^{\frac{4}{3}} 2$, that is, from D to C, we shall reach the same point C to which the line AC or -2 extends.

2. To find the imaginary roots of a number in this binomial form requires a knowledge of Trigonometry. The student who is not acquainted with the fundamental principles of that subject will not fully understand Arts. 301–305. It will, however, be for his advantage to read them.

301. The process of finding imaginary roots is the same as finding the *base* and *perpendicular* of a right-angled triangle, when the *hypothenuse* and *angles* are given.

The *hypothenuse* is the *real root* of the quantity, obtained in the usual way by evolution, and the *angle* at the base is found by the sign of the root.

302. Since the sign $-$ represents an angle of 180°, we have for these angles, for the roots of positive quantities,

$$\frac{0^\circ}{n}, \quad \frac{360^\circ}{n}, \quad \frac{2\cdot 360^\circ}{n}, \quad \frac{3\cdot 360^\circ}{n}, \quad \text{etc.};$$

and for the roots of negative quantities,

$$\frac{180^\circ}{n}, \quad \frac{3\cdot 180^\circ}{n}, \quad \frac{5\cdot 180^\circ}{n}, \quad \frac{7\cdot 180^\circ}{n}, \quad \text{etc.} \quad \text{(Art. 283.)}$$

NOTE.—Any one of these expressions which represents a multiple of 180° gives a *real root*. The rest are *imaginary*.

Whatever be the value of n, the angles less than 180° will correspond to those greater than 180°. That is, if we have an angle equal to $180° - \beta°$, there will be another, $180° + \beta°$. Hence the roots will always be found in pairs of the form

$$a \pm \sqrt{-b}.$$

303. The two roots having these relations to each other are called ***Conjugate Roots.*** The formula for these, by Trigonometry, will be

$$r\,(\cos\theta \pm \sqrt{-\sin^2\theta}),$$

in which r is the real root of the quantity, and θ the angle indicated by the sign.

EXAMPLES.

1. Find the imaginary values of $32^{\frac{1}{5}}$.

Here $n = 5$, $\theta = 72°$ and $144°$, and $r = 2$.

By the formula the imaginary roots are

$$2\,(\cos 72° \pm \sqrt{-\sin^2 72°}),$$
and
$$2\,(\cos 144° \pm \sqrt{-\sin^2 144°}).$$

Introducing approximate values for sine and cosine of $72°$ and $144°$ we have

$$2\,(.309 \pm \sqrt{-.951^2}),$$
and
$$2\,(-.809 \pm \sqrt{-.588^2}).$$

2. Find the imaginary values of $(-32)^{\frac{1}{5}}$.

Here $n = 5$, $r = 2$, and $\theta = 36°$ and $108°$.

Hence the imaginary roots are

$$2\,(\cos 36° \pm \sqrt{-\sin^2 36°}),$$
and
$$2\,(\cos 108° \pm \sqrt{-\sin^2 108°}).$$

Or, substituting values of sine and cosine of $36°$ and $108°$,

$$2\,(.809 \pm \sqrt{-.588^2}),$$
and
$$2\,(-.309 \pm \sqrt{-.951^2}).$$

NOTE.—Hence it appears that the imaginary fifth roots of 32 and of -32 are the same, except in the sign of the real term.

3. Find the imaginary values of $(729)^{\frac{1}{6}}$ and $(-729)^{\frac{1}{6}}$.

For the first, $n = 6$, $r = 3$, $\theta = 60°$ and $120°$.

The values are therefore

$$3\,(\cos 60° \pm \sqrt{-\sin^2 60°}),$$
and
$$3\,(\cos 120° \pm \sqrt{-\sin^2 120°}).$$
Or
$$3\,(\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{-3}),$$
and
$$3\,(-\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{-3}).$$

For the second, $n = 6$, $r = 3$, $\theta = 30°, 90°$, and $150°$.

The values are

$$3(\tfrac{1}{2}\sqrt{3} \pm \tfrac{1}{2}\sqrt{-2}),\quad 3(0 \pm \sqrt{-1}),\quad \text{and}\quad 3(-\tfrac{1}{2}\sqrt{3} \pm \tfrac{1}{2}\sqrt{-2})$$

The second pair of values reduce to $\pm -^{\frac{1}{2}}3 = -^{\frac{1}{2}}3$ or $-^{\frac{3}{2}}3$.

304. From the preceding we have a direct method of finding any power of an imaginary expression of the form $a \pm \sqrt{-b}$. Since a is the *base* and $\sqrt{b}$ the *perpendicular* of a right-angled triangle whose *hypothenuse*, with a proper sign, is equivalent to the binomial imaginary, we may operate by the following rule, in which r represents the *hypothenuse* and m the proper exponent of $-$ to give the quantity the right direction.

RULE. — I. *Change the expression to the form* $-^{m} r$. (Art. 305.)

II. *Raise the resulting monomial to the required power.* (Art. 266.)

III. *Restore the form* $a \pm \sqrt{-b}$. (Arts. 302, 303.)

305. To make the change from $a \pm \sqrt{-b}$ to $-^{m} r$ we have by Trigonometry,

$$r = \sqrt{a^2 + b};$$

$$m = \frac{\theta}{180^\circ}, \text{ in which } \theta = \cos^{-1} \frac{a}{r}.$$

The following examples will illustrate the process:

1. Find the cube root of $-4 + 4\sqrt{-3}$.

SOLUTION.

I. $$r = \sqrt{a^2 + b} = \sqrt{16 + 48} = 8;$$

$$m = \frac{\theta}{180^\circ} = \frac{\cos^{-1}(-\frac{4}{8})}{180^\circ} = \frac{120^\circ}{180^\circ} = \frac{2}{3}.$$

$$\therefore \quad -4 + 4\sqrt{-3} = -^{\frac{2}{3}} 8.$$

II. $$(-^{\frac{2}{3}} 8)^{\frac{1}{3}} = -^{\frac{2}{9}} 2.$$

III. $$-^{\frac{2}{9}} 2 = 2(\cos 40^\circ + \sqrt{-\sin^2 40^\circ})$$

$$= 2(.766 + \sqrt{-643^2}), \text{ } Ans.$$

This gives one of the cube roots of $-4 + 4\sqrt{-3}$. The other two will be found by using for $-^{\frac{2}{3}} 8$, its equal $-^{\frac{8}{3}} 8$, or $-^{\frac{14}{3}} 8$. (Art. 290.)

2. Find the 5th root of $243\left(\frac{1}{2} - \frac{1}{2}\sqrt{-3}\right)$.
3. Find one pair of the imaginary values of $64^{\frac{1}{6}}$.
4. Find the cube roots of -125.

CALCULUS OF RADICALS.

306. All mathematical operations upon *radicals* are performed by the same rules as *like* operations upon *rational quantities.* The student need only become familiar with the *use* of the *signs* and *symbols* of the algebraic language, and with the *principles* involved in the fundamental mathematical operations, and then apply them alike to *rational* and *radical,* to *real* and *imaginary* quantities.

Any attempt to make a difference in their application to *real* and *imaginary* quantities, is liable to result in confusion. One principle only *in addition* to those already given requires attention, viz.,

Quantities whose signs are neither the ***same*** *nor* ***opposite,*** *cannot be united in one term.*

Thus, $3a$ and $-^{\uparrow} 5a$

are neither *the same* nor *opposite in direction*, and therefore *cannot be united in the same term.* They can be added only by writing them with their signs; thus,

$$3a -^{\uparrow} 5a,$$

or, as it is frequently written,

$$3a + 5a\sqrt{-1},$$

in which $\sqrt{-1}$ has no force except as a *factor of direction,* 1 having no power as a factor.

Those, however, who prefer this form, can use

$$\sqrt{-1} \quad \text{for} \quad \sqrt{-} \quad \text{or} \quad -^{\uparrow},$$
$$\text{and} \quad -\sqrt{-1} \quad \text{for} \quad -\sqrt{-} \quad \text{or} \quad -^{\downarrow}.$$

It is wholly immaterial which form is used, if the student does not allow the presence of the 1 to give him the impression *that something besides a directive sign* is intended.

307. For ***Addition*** and ***Subtraction of Radicals*** we have the following

RULE.—*Change the signs of terms to be subtracted, and unite similar real terms in one; also similar imaginary terms, and connect dissimilar terms by their signs.*

NOTE.—*Changing signs* means *reversing the direction by applying a minus sign* to the term. Thus, changing the sign $-^{\frac{1}{2}}$ makes it $-^{\frac{3}{2}}$; that is, $+\sqrt{-}$ becomes $-\sqrt{-}$.

1. Simplify $ab - \sqrt{-a} + 2\sqrt{-a} + 3ab$.

   Ans. $4ab + \sqrt{-a}$.

2. Subtract $2a + 3\sqrt{-b}$ from $7a - 5\sqrt{-b}$.

   Ans. $5a - 8\sqrt{-b}$.

3. Subtract $-^{\frac{1}{3}}8$ from $-^{\frac{4}{3}}12$.

Changing sign of subtrahend, it becomes $-^{\frac{4}{3}}8$ and therefore lies in the same direction as $-^{\frac{4}{3}}12$. Adding, after changing the sign, gives $-^{\frac{4}{3}}20$, *Ans.*

308. For ***Multiplication of Radicals*** we have the following

RULE.—I. *To the product of the numerical factors annex the literal factors each with an exponent equal to the sum of its exponents in the several factors.* (Art. 127.)

II. *Give the product the sign —, with an exponent found by adding the exponents of — in the several factors, and subtracting from the sum the greatest even number.* (Art. 127.)

4. Multiply $a + \sqrt{-b}$ by $a - \sqrt{-b}$.

OPERATION.

$$\begin{array}{l}
a + \sqrt{-b} \\
\underline{a - \sqrt{-b}} \\
a^2 + a\sqrt{-b} \\
\underline{\quad - a\sqrt{-b} + b} \\
a^2 \qquad\qquad + b, \quad \textit{Ans.}
\end{array}$$

NOTE.—a^2 is positive, because every power of + is +, or because in both factors the sign — has zero for an exponent. The product of $a \times (+\sqrt{-b})$ has the sign $-^{\frac{1}{2}}$ or $+\sqrt{-}$, the sum of the exponents of — being $\frac{1}{2}$. Also, $a \times (-\sqrt{-b})$ has the sign $-^{\frac{3}{2}}$ or $-\sqrt{-}$, for the same reason; and $(+\sqrt{-b}) \times (-\sqrt{-b})$ has the sign +, for we have in one $-^{\frac{1}{2}}$ and in the other $-^{\frac{3}{2}}$, giving $-^2$ or +.

309. For ***Division of Radicals*** we have the following

RULE.—I. *Divide the numerical factor of the dividend by the numerical factor of the divisor, and annex the letters of both dividend and divisor each with an exponent found by subtracting its exponent in the divisor from its exponent in the dividend.*

II. *Give the quotient the sign —, with an exponent equal to its exponent in the dividend minus its exponent in the divisor, adding an even number sufficient to make this exponent positive.*

NOTE.—These rules apply especially to monomials, but operations upon polynomials are made up of operations on monomials.

5. Divide $12\,(-\,a^3b^5)^{\frac{1}{2}}$ by $-\,3\,(ab^3)^{\frac{1}{2}}$.

OPERATION.—Dividing 12 by 3 gives the numerical coefficient 4, to which annex the letters; thus, $4ab$.

For the exponent of a, we have $\frac{3}{2}-\frac{1}{2}=1$, and for b, $\frac{5}{2}-\frac{3}{2}=1$, both of which we omit.

For the exponent of $-$ we have $\frac{1}{2}-1=-\frac{1}{2}$, to which add 2, giving $\frac{3}{2}$. The quotient therefore is $-^{\frac{3}{2}}\,4ab$.

If in this example we write the sign without changing its exponent, it would read $-^{-\frac{1}{2}}\,4ab$. This does not differ from $-^{\frac{3}{2}}\,4ab$, except that the former supposes the revolution to be made in the opposite direction, so that $-^{-\frac{1}{2}}$ indicates the direction of a line (Fig. 1) turned downward from AB till it comes to AE; while $\top^{\frac{3}{2}}$ indicates the same direction produced by turning AB in the direction indicated by the arrows.*

310. The only reason, therefore, why we *add* to or *subtract* from the exponent of $-$ an even number, is that by so doing we express all signs by $+$, or by $-$, with an exponent *greater* than 0 and *less* than 2, and by using the binomial form for imaginary roots, we have all quantities reduced to four directions, viz., $+$, $-$, $-^{\frac{1}{2}}$, and $-^{\frac{3}{2}}$. The first two are *real*, and last two *imaginary*. The imaginary may be written

$$\sqrt{-} \qquad \text{and} \qquad -\sqrt{-},$$
$$\text{or} \qquad \sqrt{-1} \qquad \text{and} \qquad -\sqrt{-1}.$$

Perform the indicated operations in the following:

6. $2a^{\frac{1}{2}}x^{\frac{1}{3}} \times 3a^{\frac{3}{2}}x^{\frac{2}{3}} \div 6ax$.

7. $2\sqrt{ax}+3\sqrt{ax}-4\sqrt{-ax}+2\sqrt{-ax}$.

8. $\sqrt{a^2-x^2}+2\sqrt{a^2-x^2}-5\,(a^2-x^2)^{\frac{1}{2}}+4\,(a^2-x^2)^{\frac{1}{2}}$.

* This agrees with the general use of + and —, for if a *positive exponent* indicates revolution in one direction, a *negative exponent* should indicate revolution in the opposite direction.

9. $\left[(a+x)^{\frac{1}{2}} + (a-x)^{\frac{1}{2}}\right] \times \left[(a+x)^{\frac{1}{2}} - (a-x)^{\frac{1}{2}}\right]$.

10. $(a^2b^{\frac{1}{2}} + a^{\frac{1}{2}}b^2 - ab^{\frac{1}{2}} + a^{\frac{1}{2}}b) \times (a^{\frac{1}{2}}b + ab^{\frac{1}{2}})$.

11. $(a + 2a^{\frac{1}{2}}b^{\frac{1}{2}} + b) \div (a^{\frac{1}{2}} + b^{\frac{1}{2}})$.

12. $(a - b) \div (a^{\frac{1}{3}} - b^{\frac{1}{3}})$.

13. $(a^{\frac{1}{2}} - b^{\frac{1}{2}})\,(a^{\frac{1}{2}} + b^{\frac{1}{2}})$.

14. $(a^{\frac{1}{4}} - b^{\frac{1}{4}})\,(a^{\frac{1}{4}} + b^{\frac{1}{4}})\,(a^{\frac{1}{2}} + b^{\frac{1}{2}})$.

15. $(a^2 - b^2) \div (a^{\frac{1}{4}} - b^{\frac{1}{4}})$.

16. $\left\{a^{\frac{1}{2}}b^{\frac{1}{2}} + 2\,(ab)^{\frac{1}{2}} - 3ab^{\frac{1}{2}}\right\} - \left\{(ab)^{\frac{1}{2}} + 2ab^{\frac{1}{2}}\right\}$.

17. $-\sqrt{a} \times -\sqrt{-b} \times \sqrt{-c} \times -\sqrt{-d}$.

18. $(-a \pm a\sqrt{-3})^3$.

19. $(a \pm \sqrt{-x})\,(a \mp \sqrt{-x})$.

20. $(a^2 + x) \div (a - \sqrt{-x})$.

21. $(a - x) \div (\sqrt{-a} + \sqrt{-x})$.

22. $(a^3 - 2a^{\frac{5}{2}}x^{\frac{1}{2}} + a^2x)^{\frac{1}{2}}$.

23. $\sqrt{a + 2a^{\frac{1}{2}}x^{\frac{1}{2}} + x}$.

24. $\left(a^{\frac{m}{n}}b^{\frac{n}{m}} + a^{\frac{1}{m}}b^{\frac{1}{n}}\right)\left(a^{\frac{n}{m}}b^{\frac{m}{n}} + a^{\frac{1}{n}}b^{\frac{1}{m}}\right)$.

25. $(a^{2n} - b^{2n}) \div (a^n + b^n)$.

26. $(a^3 - b) \div (a - b^{\frac{1}{3}})$.

27. $(a^2 + b^3) \div (a^{\frac{2}{3}} + b)$.

28. $(a^3 - b^2) \div (a^{\frac{1}{2}} - b^{\frac{1}{3}})$.

29. $(a^4 + b^2) \div (a^{\frac{4}{5}} + b^{\frac{2}{5}})$.

30. $(a^8 - b^8) \div (a^{\frac{2}{3}} + b^{\frac{1}{2}})$.

31. $\left(-1 + \sqrt{5} + \sqrt{-10 - 2\sqrt{5}}\right)^5$.

32. $\left(-1 + \sqrt{5} - \sqrt{-10 - 2\sqrt{5}}\right)^5$.

33. $\left(-1 - \sqrt{5} + \sqrt{-10 + 2\sqrt{5}}\right)^5$.

34. $\left(-1 - \sqrt{5} - \sqrt{-10 + 2\sqrt{5}}\right)^5$.

FORMER NOTATION.

311. It has already been suggested that the notation $-^{\frac{1}{2}}$ takes the place of the more common notation $(-1)^{\frac{1}{2}}$, or $\sqrt{-1}$. To show how the two methods compare, we give the following examples in *Multiplication* and *Division* of imaginary quantities.

312. By the ***Common Method*** of notation every imaginary term is considered as the *product* of two factors, one of which is $\sqrt{-1}$, the other being a *real quantity* with the sign $+$ or $-$. In multiplying or dividing, these factors are considered separately. It is necessary, therefore, to find the various *integral powers* of $\sqrt{-1}$.

Since the *square* of any *square root* is the quantity itself, we have

$$(\sqrt{-1})^1 = \sqrt{-1};$$
$$\sqrt{-1} \times \sqrt{-1} = (\sqrt{-1})^2 = -1;$$
$$-1 \times \sqrt{-1} = (\sqrt{-1})^3 = -\sqrt{-1};$$
$$-\sqrt{-1} \times \sqrt{-1} = (\sqrt{-1})^4 = -(-1) = +1;$$
$$1 \times \sqrt{-1} = (\sqrt{-1})^5 = \sqrt{-1}.$$

It is evident that the higher powers will not give any new forms, and we have four forms only, viz.:

$$\sqrt{-1}; \quad -1; \quad -\sqrt{-1}; \quad \text{and} \quad +1;$$

corresponding to the explanation (Art. 292) and the illustration (Fig. 1).

EXAMPLES.

1. What is the product of $-\sqrt{-a}$ by $-\sqrt{-b}$?

SOLUTION.—Resolving each into factors, we have,

$$-\sqrt{-a} = -\sqrt{a}\sqrt{-1};$$
$$-\sqrt{-b} = -\sqrt{b}\sqrt{-1};$$
$$-\sqrt{a} \times -\sqrt{b} = \sqrt{ab};$$
$$\sqrt{-1} \times \sqrt{-1} = -1;$$
$$-1 \times \sqrt{ab} = -\sqrt{ab}.$$

That is, we resolve each quantity into two factors, one of which is real and the other the imaginary expression $\sqrt{-1}$, then multiply.

Let the student multiply the following, using either notation at pleasure.

2. Multiply $\sqrt{-4}$ by $-\sqrt{-3}$.

3. Multiply $3+\sqrt{-2}$ by $3-\sqrt{-2}$.

4. Multiply $\sqrt{-9}$ by $\sqrt{-16}$.

5. Multiply $\sqrt{-3}$ by $\sqrt{-24}$.

313. In division the factor $\sqrt{-1}$, found in both dividend and divisor, will cancel, and the quotient will be the quotient of the real factors.

EXAMPLES.

1. Divide $\sqrt{-a}$ by $\sqrt{-b}$.

SOLUTION.
$$\sqrt{-a}=\sqrt{a}\sqrt{-1};$$
$$\sqrt{-b}=\sqrt{b}\sqrt{-1};$$
$$\frac{\sqrt{a}\sqrt{-1}}{\sqrt{b}\sqrt{-1}}=\frac{\sqrt{a}}{\sqrt{b}}. \quad \textit{Ans.}$$

2. Divide $\sqrt{a}$ by $\sqrt{-b}$.

SOLUTION. $\sqrt{a}=-\sqrt{a}\times-1=-\sqrt{a}\sqrt{-1}\sqrt{-1};$
$$\sqrt{-b}=\sqrt{b}\sqrt{-1};$$
$$\frac{-\sqrt{a}\sqrt{-1}\sqrt{-1}}{\sqrt{b}\sqrt{-1}}=\frac{-\sqrt{a}\sqrt{-1}}{\sqrt{b}}=-\sqrt{-\frac{a}{b}}. \quad \textit{Ans.}$$

3. Divide $9\sqrt{-8}$ by $3\sqrt{-4}$.

4. Divide $4\sqrt{-3}$ by $2\sqrt{-9}$.

5. Divide $4\sqrt{-a^2}$ by $2a\sqrt{-2}$.

6. Divide $4+\sqrt{-2}$ by $2-\sqrt{-2}$.

REDUCTION OF RADICALS.

314. Radical expressions may frequently be simplified:

1st. By removing or introducing rational factors.
2d. By reducing radicals to like indices.
3d. By rationalizing one term of a radical fraction.

315. To Remove Rational Factors.

RULE. — *Separate the quantity under the radical sign into rational and irrational factors, and taking the required root of the rational factors, write them outside the radical sign.*

1. Given $(25a^3bx^2)^{\frac{1}{2}} + (9a^3bx^2)^{\frac{1}{2}} - (16a^3bx^2)^{\frac{1}{2}}$, to simplify the expression.

OPERATION.

$$(25a^3bx^2)^{\frac{1}{2}} = (25a^2x^2 \cdot ab)^{\frac{1}{2}} = (25a^2x^2)^{\frac{1}{2}}\,(ab)^{\frac{1}{2}} = 5ax\,(ab)^{\frac{1}{2}};$$
$$(9a^3bx^2)^{\frac{1}{2}} = (9a^2x^2)^{\frac{1}{2}}\,(ab)^{\frac{1}{2}} = 3ax\,(ab)^{\frac{1}{2}}$$
$$(16a^3bx^2)^{\frac{1}{2}} = 4ax\,(ab)^{\frac{1}{2}}.$$
$$\therefore\ (25a^3bx^2)^{\frac{1}{2}} + (9a^3bx^2)^{\frac{1}{2}} - (16a^3bx^2)^{\frac{1}{2}} =$$
$$5ax\,(ab)^{\frac{1}{2}} + 3ax\,(ab)^{\frac{1}{2}} - 4ax\,(ab)^{\frac{1}{2}} = 4ax\,(ab)^{\frac{1}{2}},\ \textit{Ans.}$$

2. Simplify $(a^3x^2 - a^2x^3)^{\frac{1}{2}}$.

OPERATION.

$$(a^3x^2 - a^2x^3)^{\frac{1}{2}} = (a^2x^2)^{\frac{1}{2}}\,(a-x)^{\frac{1}{2}} = ax\,(a-x)^{\frac{1}{2}},\ \textit{Ans.}$$

3. Simplify $\sqrt[3]{81}$.

OPERATION.

$$\sqrt[3]{81} = \sqrt[3]{27 \times 3} = \sqrt[3]{27}\,\sqrt[3]{3};\quad \therefore\ \sqrt[3]{81} = 3\sqrt[3]{3},\ \textit{Ans.}$$

4. Simplify $(a^7b^5x^2 - a^5b^7x)^{\frac{1}{3}}$.

OPERATION.

$$(a^7b^5x^2 - a^5b^7x)^{\frac{1}{3}} = (a^3b^3)^{\frac{1}{3}}\,(a^4b^2x^2 - a^2b^4x)^{\frac{1}{3}} = ab\,(a^4b^2x^2 - a^2b^4x)^{\frac{1}{3}},\ \textit{Ans.}$$

316. To Place a Rational Factor under a Radical Sign or Fractional Exponent.

RULE.—*Involve the rational factor to a power indicated by the reciprocal of the fractional exponent, and combine it with the radical factors, if there be any.*

5. Reduce a^2bc to a radical of the second degree.

OPERATION.

$$a^2bc = (a^2bc)^{\frac{1}{1}} = [(a^2bc)^2]^{\frac{1}{2}} = (a^4b^2c^2)^{\frac{1}{2}}, \quad \textit{Ans.}$$

In this case there were no radical factors with which to combine the rational factors.

6. Reduce $ab\,(ax - x^2)^{\frac{1}{2}}$ entirely to a radical form.

OPERATION.

$$ab = (a^2b^2)^{\frac{1}{2}}.$$

$$\therefore \quad ab\,(ax - x^2)^{\frac{1}{2}} = [a^2b^2\,(ax - x^2)]^{\frac{1}{2}} = (a^3b^2x - a^2b^2x^2)^{\frac{1}{2}}, \quad \textit{Ans.}$$

317. To Reduce Radicals to Like Indices.

RULE.—*Reduce their exponents to a common denominator.*

7. Reduce $a^{\frac{2}{3}} \times a^{\frac{3}{2}} \times a^{\frac{5}{6}}$ to the simplest form.

OPERATION.

$$a^{\frac{2}{3}} = a^{\frac{4}{6}}, \qquad a^{\frac{3}{2}} = a^{\frac{9}{6}}, \qquad a^{\frac{5}{6}} = a^{\frac{5}{6}}.$$

$$\therefore \quad a^{\frac{4}{6}}a^{\frac{9}{6}}a^{\frac{5}{6}} = a^{\frac{18}{6}} = a^3.$$

318. To Rationalize one Term of a Fraction.

RULE.—*Multiply both numerator and denominator by that factor which will render the required term rational.*

8. Rationalize the denominator of $\sqrt{\frac{1}{2}}$.

OPERATION.

$$\sqrt{\tfrac{1}{2}} = \sqrt{\tfrac{2}{4}} = \sqrt{\tfrac{1}{4} \times 2} = \tfrac{1}{2}\sqrt{2}, \quad \textit{Ans.}$$

9. Rationalize the denominator of $\dfrac{\sqrt{a+x}}{\sqrt{a-x}}$.

OPERATION.—Multiply both terms of the fraction by $\sqrt{a-x}$, which will give

$$\frac{\sqrt{a^2-x^2}}{a-x} = \frac{1}{a-x}\sqrt{a^2-x^2}, \quad \textit{Ans.}$$

10. Rationalize the denominator of $\dfrac{(a+x)^{\frac{1}{2}}+(a-x)^{\frac{1}{2}}}{(a+x)^{\frac{1}{2}}-(a-x)^{\frac{1}{2}}}$.

OPERATION.

Multiply by the numerator, and we have

$$\frac{[(a+x)^{\frac{1}{2}}+(a-x)^{\frac{1}{2}}]^2}{(a+x)-(a-x)} = \frac{a+x+2(a+x)^{\frac{1}{2}}(a-x)^{\frac{1}{2}}+a-x}{2x}$$

$$= \frac{a+(a^2-x^2)^{\frac{1}{2}}}{x}, \quad \textit{Ans.}$$

EXAMPLES.

Simplify the following:

1. $a - \left[5b - \left\{a - 3(c-b) + 2\left(c - \dfrac{a-2b-c}{2}\right)\right\}\right]^{\frac{1}{2}}$.

2. $\dfrac{1}{4} \cdot \dfrac{\sqrt{a^4-b^4}}{a^2-b^2} \cdot \dfrac{4a}{b} \cdot \left\{\dfrac{1-\dfrac{b^2}{a^2}}{1+\dfrac{a^2}{b^2}}\right\}^{\frac{1}{2}}$.

3. $\dfrac{1+x}{1+\sqrt{1+x}} - \dfrac{1-x}{1-\sqrt{1+x}}$.

4. $\left(\dfrac{a^2x+2ax^2+x^3}{a^2-2ax+x^2}\right)^{\frac{1}{2}} - \left(\dfrac{a^2x-2ax^2+x^3}{a^2+2ax+x^2}\right)^{\frac{1}{2}}$.

5. $(5+2\sqrt{3})(5-2\sqrt{3})$.

6. $\dfrac{8\sqrt{2}+12\sqrt{6}-4\sqrt{10}}{4\sqrt{2}}$.

7. $\dfrac{4-\sqrt{-2}}{-2\sqrt{2}}$.

8. $(1 \pm \sqrt{-3})^3 + (-1 \pm \sqrt{-3})^3$.

9. $\dfrac{a}{a+\sqrt{a^2-b^2}} + \dfrac{a}{a-\sqrt{a^2-b^2}}$.

10. $\dfrac{(a+x)^{\frac{1}{2}}+(a-x)^{\frac{1}{2}}}{(a+x)^{\frac{1}{2}}-(a-x)^{\frac{1}{2}}} + \dfrac{(a+x)^{\frac{1}{2}}-(a-x)^{\frac{1}{2}}}{(a+x)^{\frac{1}{2}}+(a-x)^{\frac{1}{2}}}$.

11. $\dfrac{-5\sqrt{-3}}{10\sqrt{-2}} + \dfrac{10\sqrt{-3}}{-5\sqrt{-2}} + \dfrac{5\sqrt{-3}}{2\sqrt{-2}}$.

12. $\dfrac{-5\sqrt{-a}}{3\sqrt{-b}} \times \dfrac{2\sqrt{b}}{\sqrt{a}} \times \sqrt{-\dfrac{3}{10}}$.

13. $4\sqrt[4]{-2} \times -3\sqrt[4]{-5} \times \frac{1}{2}\sqrt[4]{-3} \div 3\sqrt[4]{-30}$.

14. $$\dfrac{\dfrac{(1+\sqrt{x})^{\frac{1}{2}}\left(-\dfrac{1}{2\sqrt{x}}\right)}{2(1-\sqrt{x})^{\frac{1}{2}}} - \dfrac{(1-\sqrt{x})^{\frac{1}{2}}\cdot\dfrac{1}{2\sqrt{x}}}{2(1+\sqrt{x})^{\frac{1}{2}}}}{(1+x^{\frac{1}{2}})^2-(1-x^{\frac{1}{2}})^2}.$$

15. $$\dfrac{\left\{(1+x)^{\frac{1}{2}}-(1-x)^{\frac{1}{2}}\right\}\left\{\dfrac{1}{2(1+x)^{\frac{1}{2}}}+\dfrac{1}{2(1-x)^{\frac{1}{2}}}\right\}}{\left[\left\{(1+x)^{\frac{1}{2}}-(1-x)^{\frac{1}{2}}\right\}^2-2\right]^{-1}}$$
$$+\dfrac{\left\{(1+x)^{\frac{1}{2}}+(1-x)^{\frac{1}{2}}\right\}\left\{\dfrac{1}{2(1+x)^{\frac{1}{2}}}-\dfrac{1}{2(1-x)^{\frac{1}{2}}}\right\}}{\left[\left\{(1+x)^{\frac{1}{2}}-(1-x)^{\frac{1}{2}}\right\}^2-2\right]^{-1}}.$$

16. $$\dfrac{1+\dfrac{x}{(1+x^2)^{\frac{1}{2}}}}{2\sqrt{x+\sqrt{1+x^2}}}.$$

17. $\sqrt{a^3x^2-a^2x^3} + \sqrt{a^5x^4-a^4x^5} + \sqrt{(a-x)^3}$.

18. $\dfrac{x}{x-a} - \dfrac{x}{x+a} - \dfrac{\dfrac{x+a}{x-a} - \dfrac{x-a}{x+a}}{\dfrac{x+a}{x-a} + \dfrac{x-a}{x+a}}.$

19. $(a^5 - x^5 - 5a^4x + 5ax^4 + 10a^3x^2 - 10a^2x^3)^{\frac{1}{5}}.$

20. $(a^3 - 3a^2b - 12abx + 6a^2x + 12ax^2 + 3ab^2 + 8x^3$
$- 12bx^2 + 6b^2x - b^3)^{\frac{1}{3}}.$

21. $\sqrt{-ac} \times -\sqrt{bc} \times -\sqrt{-ac} \times -\sqrt{-abc}.$

22. $\dfrac{a(a+b)^{-1} + b(a-b)^{-1}}{a(a-b)^{-1} - b(a+b)^{-1}}.$

23. $a^{\frac{1}{2}}b^{-\frac{2}{3}}c^{\frac{1}{4}} \times a^{\frac{2}{3}}b^{\frac{1}{2}}c^{-\frac{1}{3}} \times a^{-\frac{2}{3}}b^{-\frac{1}{4}}c^{\frac{2}{3}} \times a^{\frac{1}{3}}b^{\frac{1}{2}}c.$

24. $a^{\frac{1}{m}}b^{\frac{1}{n}}c^{s} \times a^{-\frac{1}{m}}b^{-\frac{1}{n}}c^{-s} \times a^{m}b^{n}c^{s} \times a^{n}b^{m}c^{2}.$

25. $a^{\frac{m+n}{n}}\,b^{\frac{m+n}{m}}\,c^{mn} \times a^{\frac{m+n}{m}}\,b^{\frac{m+n}{n}}\,c^{\frac{m}{n}}.$

26. $\dfrac{a+\sqrt{-b}}{a^{-\frac{2}{3}}b^{\frac{1}{2}}} \times \dfrac{a-\sqrt{-b}}{a^{-\frac{1}{3}}b^{\frac{1}{2}}}.$

Rationalize the denominators of

27. $\dfrac{a}{\sqrt{a}-\sqrt{b}}.$

28. $\dfrac{1}{x+y^{\frac{1}{3}}}.$

29. $\dfrac{b^3}{b^{\frac{1}{3}}-a^{\frac{1}{3}}}.$

30. $\dfrac{x^2}{x^{\frac{1}{2}}+y^{\frac{1}{4}}}.$

31. $\dfrac{x^2}{x^{\frac{1}{3}}-y^{\frac{1}{4}}}.$

32. $\dfrac{1}{x^{\frac{2}{3}}-y^{\frac{1}{3}}}.$

33. $\dfrac{1}{x^{\frac{2}{3}}+y^{\frac{2}{3}}}.$

34. $\dfrac{1}{x^{\frac{1}{2}}-y^{-\frac{1}{3}}}.$

35. $\dfrac{1}{a^{-\frac{1}{2}}+y^{-\frac{1}{3}}}.$

NOTE.—For the last nine examples see Art. 155, and examples 26–30, page 123.

RADICAL EQUATIONS.

319. Equations containing ***Radicals*** may often be reduced to *simple equations* by the following

RULE.—*Involve both members to a power indicated by the reciprocal of the radical exponent.*

NOTES.—1. Before *involving* the quantities, it is generally best to clear of fractions, and transpose the terms, so that one member of the equation shall contain but one radical term.

2. In reducing such equations, the student should remember that any of the methods of reduction or rationalization of radicals may be employed, in accordance with Axiom 1. (Art. 38.)

1. Given $(12 + x)^{\frac{1}{2}} = 2 + \sqrt{x}$, to find x.

OPERATION.

Squaring,	$12 + x = 4 + 4\sqrt{x} + x.$
Transposing and uniting terms,	$4\sqrt{x} = 8.$
Dividing by 4, and squaring,	$x = 4$, *Ans.*

2. $(2x + 3)^{\frac{1}{3}} + 4 = 7.$

3. $\sqrt{x - 16} = 8 - \sqrt{x}.$

4. $\sqrt{4a + x} = 2(b + x)^{\frac{1}{2}} - \sqrt{x}.$

5. $\dfrac{\sqrt{x} + 2a}{\sqrt{x} + b} = \dfrac{\sqrt{x} + 4a}{\sqrt{x} + 3b}.$

6. $\dfrac{3x - 1}{\sqrt{3x} + 1} = 1 + \dfrac{\sqrt{3x} - 1}{2}.$

7. $\dfrac{\sqrt{mx} - \sqrt{m}}{\sqrt{cx} - \sqrt{c}} = \dfrac{\sqrt{x} + m}{\sqrt{x} + c}.$

8. $(5 + x)^{\frac{1}{3}} + \sqrt[3]{5 - x} = \sqrt[3]{10}.$

9. $\sqrt{x + x^{\frac{1}{2}}} - \sqrt{x - x^{\frac{1}{2}}} = \frac{3}{2}\left(\dfrac{\sqrt{x}}{1 + \sqrt{x}}\right)^{\frac{1}{2}}.$

CHAPTER XI.

EQUATIONS OF THE SECOND DEGREE.

320. ***Equations of the Second Degree*** are called ***Quadratics***, and may all be reduced to the general form

$$Ax^2 + Bx + C = 0, \qquad (1)$$

in which A is *positive*, and B and C are either *positive* or *negative*.

321. A cannot be 0, for the *first term* would then disappear, and the equation be of the *first degree.*

If $B = 0$, the *second term* disappears, but the equation is still of the *second degree.*

If $C = 0$, the *third term* disappears, and the equation, though still of the *second degree*, may be changed to one of the *first degree* by simply dividing by x.

322. Hence we have but *two cases of equations of the second degree*, represented by

$$Ax^2 + C = 0,$$

which is called the *incomplete equation* of the second degree, or the *pure quadratic;* and

$$Ax^2 + Bx + C = 0,$$

called the *complete equation* of the second degree or the *affected quadratic.*

323. A ***Complete Equation*** is one in which the series of powers of the unknown quantity is *complete,* from the highest to the lowest; as,

$$ax^3 - bx^2 + cx + d = 0,$$

in which the exponents of the powers of x form an unbroken series, from 3 to 0.

324. An ***Incomplete Equation*** is one in which one or more terms of this series are wanting; as,

$$ax^3 + bx + c = 0,$$

in which the term containing x^2 is wanting.

325. Dividing equation (1) by A, we have

$$x^2 + \frac{B}{A}x + \frac{C}{A} = 0. \qquad (2)$$

Substituting for $\frac{B}{A}$ and $\frac{C}{A}$, $2a$ and b,

$$x^2 + 2ax + b = 0, \qquad (3)$$

a form to which every *quadratic* may be reduced, in which a and b represent *any quantities whatever, integral or fractional, positive or negative,* and a may be zero.

326. To reduce equation (3), if $b = a^2$, we may take the square root of both members (Art. 156, Cor. 1), which will give

$$x + a = \pm 0,$$

and $$x = -a \pm 0.$$

If $b \gtrless a^2$, we may always add to b such a quantity as will make it equal to a^2. This quantity will be $a^2 - b$. (Art. 112, 4°.) But to preserve the *equality,* the same quantity must be *added* to both members (Art. 38, Ax. 1), which will give

$$x^2 + 2ax + a^2 = a^2 - b. \qquad (4)$$

327. This is called ***Completing the Square,*** by which is meant

Making the first member a ***Perfect Square.***

328. It will be observed that the addition of $a^2 - b$ to both members is the same as transposing b and adding a^2 to both members. Hence,

To make the first member of a quadratic a *perfect square,* we have this

RULE.—*Transpose the absolute term, and add to both members the square of half the coefficient of x.*

329. Taking the square root of equation (4),

$$x + a = \pm \sqrt{a^2 - b},$$

$$\text{and} \qquad x = -a \pm \sqrt{a^2 - b}. \qquad \text{(A)}$$

If $a = 0$, the equation is *incomplete,* and (A) becomes

$$x = \pm \sqrt{-b}. \qquad \text{(B)}$$

As equation (A) is general, applying to *all equations of the second degree, complete and incomplete,* its discussion will furnish all the principles relating to the roots of such equations. These may be enumerated as follows :

1°. *The equation has two roots* (Art. 232),

$$-a + \sqrt{a^2 - b} \qquad \text{and} \qquad -a - \sqrt{a^2 - b}.$$

2°. *The sum of the roots equals $-2a$, or the coefficient of x first power with its sign changed.* Hence,

3°. *The sign of the numerically greater root will be unlike the sign of the second term or $2ax$.* (Art. 106.)

4°. *The product of the roots equals b, or the coefficient of x zero power.* Hence,

5°. *The roots will have like signs when b is positive, and unlike signs when b is negative.* (Art. 130.)

6°. *The roots will be equal when $b = a^2$, and numerically equal with opposite signs when $a = 0$.*

7°. *The roots will be real when b is not greater than a^2, and imaginary when b is greater than a^2.*

8°. *The first member of $x^2 + 2ax + b = 0$ is the product of $(x - r)(x - r')$, r and r' representing the roots.*

NOTE.—The student will easily find the proof of these propositions by an examination of the roots $-a + \sqrt{a^2 - b}$ and $-a - \sqrt{a^2 - b}$.

330. By (8°) we have a ready method of forming an equation having any given roots.

For example, to form an equation whose roots shall be 3 and -5, we have only to multiply $x - 3$ by $x - (-5)$; thus,

$$(x - 3)(x + 5) = x^2 + 2x - 15.$$

This put equal to zero is the equation sought.

Find equations having the following roots:

1. 7 and -2.
2. a and b.
3. a and $-b$.
4. $-a$ and $-b$.
5. $3 \pm \sqrt{-3}$.
6. $-2 \pm \sqrt{-1}$.

331. From the same we have also an easy method of factoring a function of x of the second degree, by making the function $= 0$, and finding the values of x. These values connected separately with x by the sign $-$ will give the factors sought.

332. By (5°) it appears that both roots will be *real*, or both *imaginary*. It will be shown hereafter that an equation with real coefficients cannot have an odd number of imaginary roots.

NOTE.—Equation (A) should be regarded as a *formula* by which the roots of a quadratic can be written without writing out the process of completing the square. The student should commit it to memory for this purpose.

333. The formula applies equally to the complete and incomplete equation, reducing in the latter case to formula (B).

334. When the reduction of the equation to the form

$$x^2 + 2ax + b = 0$$

involves the use of fractions, this may be avoided by the following method.

Let the equation be reduced to the form

$$ax^2 + bx + c = 0.$$

Multiplying by $4a$, $\quad 4a^2x^2 + 4abx + 4ac = 0.$

Adding $b^2 - 4ac$, $\quad 4a^2x^2 + 4abx + b^2 = b^2 - 4ac.$

Extracting the square root,

$$2ax + b = \pm\sqrt{b^2 - 4ac},$$

and

$$x = -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac}. \qquad \text{(C)}$$

Hence we have for completing the square another

RULE.—I. *Reduce the equation to the form* $ax^2 + bx + c = 0$.

II. *Transpose the absolute term, and multiply by* 4 *times the coefficient of* x^2.

III. *Add to both members the square of the original coefficient of* x.

NOTE.—By remembering formula (C), it may be used instead of formula (A); but it is better to reduce all quadratics to the form (3), and to use formula (A).

335. After reducing the following equations to the form (3), let the student answer by inspection the following questions:

1st. What is the sum and what the product of the roots?
2d. Are the roots equal or unequal?
3d. Are they real or imaginary?
4th. Have they like or unlike signs?
5th. If like, what is the sign? If unlike, what is the sign of the greater?

NOTE.—The student should also practice the substitution of the roots obtained to verify the reduction. (Art. 233.)

Reduce the following:

1. $7x^2 - 16x + 68 = (4x - 2)^2.$

2. $(x + 5)^2 + (x - 5)^2 = 68.$

3. $(x - a)^2 + (x + b)^2 = 0.$

4. $\dfrac{x+2}{x-1} - \dfrac{4-x}{2x} = \dfrac{7}{3}.$

5. $\dfrac{x-6}{x+2} - \dfrac{3-x}{2+x} = \dfrac{x}{3}.$

6. $\dfrac{3}{x-3} - \dfrac{2}{x-2} = \dfrac{5}{x+1}.$

7. $\dfrac{x}{x+2} + \dfrac{x}{x-2} = \dfrac{1}{x^2-4}.$

8. $\dfrac{1}{x-a} + \dfrac{1}{x-b} + \dfrac{1}{x-c} = 0.$

9. $\dfrac{x+3}{x-3} + \dfrac{x-3}{x+3} = \dfrac{17}{4}.$

10. $\dfrac{x}{x+1} + \dfrac{x+1}{x} = \dfrac{5}{2}.$

11. $\dfrac{1}{(x-b)(x-c)} + \dfrac{1}{(x+b)(x+c)} = 0.$

12. $(x-1)(x-2) + (x-2)(x-3) = (x-3)(x-4)$

13. $x[x - (a+b)] - x[a - (b+x)] = \dfrac{b}{2}.$

14. $\dfrac{1}{x+1} + \dfrac{2}{x-2} + \dfrac{3}{x+3} = \dfrac{11}{x+1}.$

15. $\dfrac{x^2}{2} - \dfrac{x}{3} + 7\tfrac{3}{8} = 8.$

16. $x^2\left(1 - \dfrac{1}{x}\right) = 8(x+2).$

17. $\dfrac{a-b}{c}x + \dfrac{3x^2}{2} - \dfrac{a^2}{c^2} = \dfrac{b+a}{c}x + \dfrac{x^2}{2} - \dfrac{b^2}{c^2}.$

18. $a^2 + b^2 - 2bx + x^2 = \dfrac{m^2x^2}{n^2}.$

19. $mx^2 + mn = 2mx\sqrt{n} + na^2.$

PROBLEMS.

1. Divide 21 into two parts, such that the square of the less shall be $\frac{4}{25}$ of the square of the greater.

2. Divide 14 into two parts, such that their product shall be $\frac{4}{10}$ of the square of the greater.

3. The sum of the squares of two numbers is 370, and the difference of their squares 208. What are the numbers?

4. The sum of the squares of two numbers is a^2, and the difference of their squares is b^2. What are the numbers?

5. Find a number such that when divided by the product of its digits, the quotient will be 2, and the sum of its digits is 9.

6. Divide 95 into two parts whose product is 2146.

7. The sum of two numbers is 100, and the difference of their square roots is 2. What are the numbers?

8. A man travelled 60 miles, and if he had travelled 1 mile an hour more, he would have required 3 hours less to perform the journey. At what rate did he travel?

9. A boy bought oranges for 30 cents. If he had bought 5 more for the same sum, they would have cost him 1 cent apiece less. How many did he buy?

10. Divide a into two parts, such that the sum of their square roots shall be s.

11. A and B were travelling the same road at the same rate. At a certain point O, A overtook C, who was travelling at the rate of 3 miles in 2 hours, and two hours later met D, travelling 9 miles in 4 hours. B overtook C 5 miles from O, and met D 40 minutes before he was 19 miles from O. Where was B when A was 50 miles from O?

12. A went from C to D, travelling a miles an hour. When he was b miles from C, B started from D towards C, and went every hour $\frac{1}{n}$ of the distance from D to C. When B had travelled as many hours as he went miles an hour, he met A. Find the distance from C to D.

13. From 1850 to 1860, the population of a certain town increased 1200, which was a percentage of gain 50 greater than for the previous decade. From 1860 to 1870, the increase was $\frac{1}{2}$ of the whole gain from 1840 to 1860, and the increase from 1840 to 1870 was 6%. What was the population in 1840, 1850, 1860, and 1870?

14. Divide 30 into two parts, such that the product of their squares shall be 46656.

15. A body of men were formed into a hollow square 3 deep, when it was observed that with an addition of 52 to their number a solid square might be formed, of which the number of men in each side would be greater by 2 than the square root of the number of men in the hollow square. What was the number of men in the hollow square?

16. A looking-glass 12 by 18 inches has a frame of uniform width, and of the same area as the glass. What is the width of the frame?

17. A man being asked his own age and that of his son, answered: "If you add to my age twice the square root of itself and subtract 24, the remainder will be nothing. The same is also true of my son's age." What was the age of each?

18. The product of two numbers is m, and the difference of their cubes is equal to n times the cube of their difference. What are the numbers?

19. A company at a hotel had a bill of $17.50 to pay, but before it was paid two of them left, when those who remained had each to pay $1 more. How many were in the company at first?

20. A man bought a certain number of sheep for $300, out of which he reserved 15, and sold the remainder for $270, gaining 50 cents a head. How many sheep did he buy, and at what price?

21. A square courtyard has a rectangular walk around it. The side of the court is 2 yards less than 6 times the breadth of the walk; and the number of square yards in the walk exceeds the number of yards in the perimeter of the court by 92. Find the area of the court.

HIGHER EQUATIONS SOLVED BY QUADRATICS.

336. Many equations of a higher degree than the second may be reduced as quadratics. These may all be reduced to the form

$$x^{2n} + 2ax^n + b = 0,$$

in which n is any number, *integral* or *fractional*.

By completing the square, we have

$$x^{2n} + 2ax^n + a^2 = a^2 - b.$$

Taking the root, $\quad x^n + a = \pm \sqrt{a^2 - b}.$

$$x^n = -a \pm \sqrt{a^2 - b}$$

and $\quad x = (-a \pm \sqrt{a^2 - b})^{\frac{1}{n}}. \qquad$ (A′)

We see that formula (A) may be used for writing the values of x^n, from which the roots are obtained by taking the n^{th} root.

Thus, $x + 4x^{\frac{1}{2}} - 12 = 0$, in which $n = \frac{1}{2}$, gives

$$x^{\frac{1}{2}} = -2 \pm \sqrt{12 + 4} = -2 \pm 4 = 2 \text{ or } -6.$$

$$x = 4 \text{ or } 36, \quad \textit{Ans.}$$

337. Equations of this sort should have $2n$ roots (Art. 232), and this we see is true, for x^n has two values, and x is the n^{th} root of these values. Therefore, as there are n n^{th} roots of a quantity (Art. 290), x has $2n$ values.

338. To apply this to the last equation, it will be observed that it is of the *second degree* with respect to $x^{\frac{1}{2}}$, and $x^{\frac{1}{2}}$ has *two values;* but with respect to x it is of the $\frac{2}{2}$ *degree.* The *degree units* are, so to speak, only *half-units;* that is, the successive exponents differ by $\frac{1}{2}$ instead of 1.

So also the roots 4 and 36 are only *half-roots;* that is, each root may with mathematical accuracy be substituted in *two ways,* while only *one* of these will satisfy the equation.

Thus, substituting 4 for x, we have

$$4 \pm 4 \cdot 2 = 12.$$

We get the double sign for the second term, because the square root of 4 is ± 2, but the sign $+$ only will satisfy the equation.

If we substitute 36, we have

$$36 \pm 4 \cdot 6 = 12,$$

in which only the sign $-$ will give a true result. Hence we see that this equation of the $\frac{2}{2}$ degree has two *half-roots.*

339. In the form $x^{2n} + 2ax^n + b = 0$, x may be represented by a *binomial* or a *polynomial;* as,

$$(2x^2 - 1)^2 - 4\,(2x^2 - 1) - 21 = 0.$$

If we substitute y for $2x^2 - 1$, we get

$$y^2 + 4y - 21 = 0,$$

from which we may obtain y and afterwards x; but it is better to consider $2x^2 - 1$ the unknown quantity, and save the labor of making the substitution. By so doing, we get directly from the formula,

$$2x^2 - 1 = 2 \pm \sqrt{21 + 4} = 2 \pm 5 = 7 \text{ or } -3.$$

$$x^2 = 4 \text{ or } -1.$$

$$x = \pm 2 \text{ or } \pm \sqrt{-1}.$$

Two of the values of x are real and two imaginary.

EXAMPLES.

1. $x^{\frac{1}{n}} - x^{\frac{2}{n}} = a.$
2. $x^{\frac{1}{n}} + x^{\frac{1}{2n}} = b.$
3. $\sqrt{a^2 - x^2} + 2\,(a^2 - x^2) = m + n.$
4. $\dfrac{1}{1 + (a - x)^{\frac{1}{2}}} - \dfrac{1}{\sqrt{1 + (a - x)^{\frac{1}{2}}}} = \dfrac{1}{b}.$
5. $x^{\frac{1}{3}} + x^{\frac{2}{3}} = 12.$
6. $x^{\frac{1}{3}} + x^{\frac{1}{6}} = 6.$
7. $\left(\dfrac{x}{x + 1}\right)^2 + \left(\dfrac{x + 1}{x}\right)^{-1} = \dfrac{n}{(n - 1)^{-1}}.$

8. $x - \dfrac{1}{(x+5)^{-\frac{1}{2}}} = 1.$

9. $1 + \left(\dfrac{x-a}{x}\right)^{\frac{1}{2}} = \left(\dfrac{x}{a+x}\right)^{-\frac{1}{2}}.$

10. $\dfrac{x^2+a}{(x+b)^{-1}} = \dfrac{a}{b^{-1}}.$

11. $(3x+1)^{\frac{1}{2}} + (3x+1)^{\frac{1}{4}} = 6.$

12. $\sqrt{1+x}\,(1-\sqrt{1-x}) = \sqrt{1-x}\,(1+\sqrt{1+x}).$

13. $\dfrac{x-\sqrt{x+1}}{x+\sqrt{x+1}} = \dfrac{1}{5}.$

14. $(a^{\frac{1}{2}} + x^{\frac{1}{2}})^{\frac{1}{3}} = (a^{\frac{1}{3}} + x^{\frac{1}{3}})^{\frac{1}{2}}.$

15. $x^4 - 4x^3 + 5x^2 - 2x = 0.$

16. $(1 - x + x^2)^{\frac{1}{2}} - (1 + x + x^2)^{\frac{1}{2}} = (1 - 3^{\frac{1}{2}})(1 + 3^{\frac{1}{2}}).$

17. $x^3 + 1 = 0.$

18. $x^6 - 1 = 0.$

19. $\dfrac{x^2-a^2}{x^2+a^2} + \left(\dfrac{x^2-a^2}{x^2+a^2}\right)^{-1} = \dfrac{41}{20}.$

20. $1 + (x-1)^{\frac{1}{2}} - [1 + (x+1)^{\frac{1}{2}}] = 1 - \sqrt{3}.$

21. $\dfrac{x^{\frac{1}{2}} + (x-a)^{\frac{1}{2}}}{x^{\frac{1}{2}} - (x-a)^{\frac{1}{2}}} = \dfrac{n^2a}{x-a}.$

22. $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}} = b.$

23. $\dfrac{1}{1+\sqrt{1-x^2}} - \dfrac{1}{1-\sqrt{1-x^2}} = \dfrac{\sqrt{3}}{x^2}.$

THE PROBLEM OF THE LIGHTS.

340. Two lights, A and B, of given intensities, are situated a given distance apart. Find the point on the line AB where the lights will give equal illumination.

Let u_1 and u_2 = the illumination from A and B respectively, at any distance x.

Then, since by the principles of optics the illumination varies inversely as the square of the distance (Arts. 375–380),

$$u_1 = \frac{m}{x^2} \quad \text{and} \quad u_2 = \frac{n}{x^2},$$

in which m and n are constants, depending on the intensities of the lights. If we make $x = 1$ in each of these equations, we have

$$u_1 = m \quad \text{and} \quad u_2 = n.$$

Hence we see that m and n represent the illumination from each light at a unit's distance.

Making x = the distance from A toward B to the point of equal illumination, and d the distance between the lights, we have for that point,

$$u_1 = \frac{m}{x^2}, \quad \text{and} \quad u_2 = \frac{n}{(d-x)^2},$$

Since, by hypothesis, u_1 and u_2 for this point are equal,

$$\frac{m}{x^2} = \frac{n}{(d-x)^2}.$$

From which we have

$$x = \frac{d\sqrt{m}}{\sqrt{m} \pm \sqrt{n}}.$$

The double sign in the denominator gives two values of x, and there are two points of equal illumination.

341. To discuss this result, assume as follows:

1st. Let $m > n$.

Then both values of x will be positive, one less and one greater than d; that is, one point will be between A and B, nearer to B, and the other beyond B. This is evidently as it should be, since the point sought must be nearer the less light.

In like manner, if $m < n$, the point between A and B comes nearer to A, and the other point without is on the side of A.

2d. Let $m = n$.

This gives the two values

$$x = \tfrac{1}{2}d \quad \text{and} \quad x = \infty.$$

The first of these is evidently correct, since if the lights are equal, the point of equal illumination ought to be equidistant from each.

The second value (∞) does not so readily appear true; but when we consider that $\infty \pm d = \infty$ (Art. 412), we find this point also equidistant from the lights.

Another question might arise here, whether the ∞ be $+$ or $-$, and if $+$ why not also $-$. This is answered by observing that while the lights differ in intensity, the point without is on the side of the less light, and as the less light becomes more nearly equal to the greater, this point recedes until, when the difference between the lights is infinitesimal, its distance is infinite, but still on the side of the light which is infinitesimally less.

3d. Let $d = 0$, and $m \gtrless n$.

Then both values of x are 0. The lights are then at the same point, and no place except this point will be equally illuminated. At this point $\left(\text{by the theory that gives } u = \frac{m}{x^2}\right)$, if $x = 0$, the illumination is *infinite*. This supposes the light to come from a mathematical point, which has *no dimensions*, a thing which never occurs, since the source of light is always some portion of matter having dimensions. Practically, therefore, no such conditions can be fulfilled; but it furnishes a good illustration of the general character of mathematical analysis, which does not stop with the possibilities of physical conditions, but gives the results which would follow from given laws, if the physical conditions could be realized.

4th. Let n be negative.

Then the values of x are imaginary, showing that with such conditions there is *no point of equal illumination*.

In (3d), if the conditions could be fulfilled, the result would be real, and the problem admits of a definite answer. In (4th), even if the conditions could be fulfilled, there would be no such point as the one sought, and the result is therefore imaginary. To fulfill the conditions of n being negative, it would require that B should diffuse in all directions some light-absorbing vapor, by which the light from A should be partially neutralized, the law of its diffusion being the same as that of the diffusion of light. But even then it is evident there would be no point equally illuminated by A and B. If m and n are both negative, the result becomes real.

5th. Let $d = 0$ and $m = n$.

Then $x = 0$ and $\frac{0}{0}$. This last value is indeterminate (Art. 210), and represents any quantity whatever, which is evidently a correct result, since equal lights situated at the same point should illuminate equally all points at whatever distance.

SIMULTANEOUS EQUATIONS OF THE SECOND DEGREE.

342. The *general* equation of the second degree between two unknown quantities is

$$ay^2 + bxy + cx^2 + dy + ex + f = 0.$$

This equation, combined for elimination with another of like form, would produce an equation of the *fourth* degree, but combined with the general equation of the first degree,

$$a'y + b'x + c' = 0,$$

gives an equation of the second degree. Hence,

343. *Simultaneous equations with two unknown quantities, one of the second and one of the first degree, can always be reduced as quadratics.*

344. There are also certain classes of simultaneous equations with two unknown quantities which may be reduced as quadratics, when both are of the second degree. These are,

1st. *Homogeneous Equations.*

2d. *Symmetrical Equations.*

345. ***Homogeneous Equations*** contain the same number of unknown factors in each term, except the *absolute* term. Such equations can always be reduced by substituting for one of the unknown quantities the product of the other by a *new* unknown factor.

346. ***Symmetrical Equations*** have the unknown quantities similarly involved; as,

$$x^2 + xy + y^2 = 19.$$

These can usually be reduced by substituting for x and y respectively the *sum* and *difference* of two other unknown quantities.

347. The following is an example of a *quadratic* and a *simple* equation:

$$1. \quad \begin{cases} x^2 + y^2 = 20 & (1) \\ x - y = 2 & (2) \end{cases}$$

From (2), $\quad x = 2 + y \quad (3)$

Substituting in (1), $\quad (2 + y)^2 + y^2 = 20 \quad (4)$

Or, $\quad y^2 + 2y = 8$

$$y = -1 \pm \sqrt{8 + 1} = -1 \pm 3$$

$$\left.\begin{aligned} y &= 2 \text{ or } -4, \\ x &= 4 \text{ or } -2. \end{aligned}\right\} \textit{Ans.}$$

Substituting in (2), (the second line above)

348. The following illustrates the reduction of *homogeneous* equations:

$$2. \quad \begin{cases} x^2 + xy = 10 & (1) \\ xy + y^2 = 15 & (2) \end{cases}$$

Put $\quad x = zy$

Then $\quad z^2y^2 + zy^2 = 10 \quad (3)$

$\quad zy^2 + y^2 = 15 \quad (4)$

From (3), $\quad y^2 = \dfrac{10}{z^2 + z} \quad (5)$

From (4), $\quad y^2 = \dfrac{15}{z + 1} \quad (6)$

Equating, $\quad \dfrac{10}{z^2 + z} = \dfrac{15}{z + 1}$

Hence, $\quad z = \frac{2}{3} \text{ or } -1$

Substituting $\frac{2}{3}$ for z in (5) or (6), we have,

$$y = \pm 3$$

$$\therefore \quad x = \pm 2$$

Substituting -1 for z in (5) or (6), we have

$$y = \pm \infty$$

$$\therefore \quad x = \mp \infty$$

Since the combination of these equations of the second degree give an equation of the fourth degree, there should be four roots. Two of them in this case are ∞. Substituting $+\infty$ for x, and $-\infty$ for y, we have,

$$\infty^2 - \infty^2 = 10$$

$$\infty^2 - \infty^2 = 15$$

results which are consistent, as will be seen, Art. 412.

349. As an example of *Symmetrical Equations*, we have,

$$3. \quad \begin{cases} x^2 + y^2 = 13 & (1) \\ \quad\;\; xy = 6 & (2) \end{cases}$$

Multiplying (2) by 2, and adding it to and subtracting it from (1),

$$x^2 + 2xy + y^2 = 25$$
$$x^2 - 2xy + y^2 = 1$$
$$\therefore \quad x + y = \pm 5$$
$$x - y = \pm 1$$

Adding, $\quad 2x = \pm 6$

Subtracting, $\quad 2y = \pm 4$

$$\left.\begin{array}{l} x = \pm 3 \\ y = \pm 2 \end{array}\right\} \textit{Ans.}$$

350. *Symmetrical Equations* with two unknown quantities may frequently be reduced, when one is of the first degree and the other of a degree higher than the second.

$$4. \quad \begin{cases} x + y = 8 & (1) \\ x^3 + y^3 = 152 & (2) \end{cases}$$

Let $\quad x = u + v$

" $\quad y = u - v$

Then $\quad x + y = 2u = 8$

And $\quad u = 4$

Substituting in (2), $\quad (4 + v)^3 + (4 - v)^3 = 152$

Developing and reducing, $\quad 24v^2 = 24$

$$v = \pm 1$$

$$x = u + v = 4 \pm 1 = 5 \text{ or } 3$$
$$y = u - v = 4 \mp 1 = 3 \text{ or } 5$$

Reduce the following by these or other methods, as shall be found feasible:

5. $x - y = 4,$
$x^3 - y^3 = 124.$

6. $x + y = 9,$
$x^2 + y^2 = 53.$

7. $x + y = a,$
$x^3 + y^3 = b.$

8. $x + y = 5,$
$x^4 + y^4 = 97.$

9. $2x^2 + xy = 14,$
$2y^2 - xy = 12.$

10. $x^3y - y = 21,$
$x^2y - xy = 6.$

11. $x^2 + 2xy + y + 3x = 73,$
$y^2 + 3y + x = 44.$

12. $x^2y^2 + 4xy = 12,$
$x + y = 1.$

13. $x - y = 2,$
$x^3 - y^3 = 56.$

14. $x^2 - y^2 = 8,$
$\sqrt{x^2 + y^2} = \dfrac{1}{10^{-\frac{1}{2}}}.$

15. $\dfrac{x + y}{x - y} + \dfrac{x - y}{x + y} = \dfrac{10}{3},$
$x^2 + y^2 = 5.$

16. $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{a},$
$x^2 + y^2 = b.$

17. $x^3y^2 + x^2y^3 = -4,$
$x^2y + xy^2 = 2.$

18. $\sqrt{x} + \sqrt{y} = 6,$
$x + y = 20.$

19. $\sqrt{x} + \sqrt{y} = 4,$
$x^{\frac{3}{2}} + y^{\frac{3}{2}} = 28.$

20. $x - y = 1,$
$x^4 - y^4 = 15.$

21. $x^2 + 3xy - y^2 = 36,$
$3x + 2y = 16.$

22. $\dfrac{x}{y} + \dfrac{y}{x} = \frac{5}{2},$
$x + xy + y = 14.$

23. $x^2 + xy = 40,$
$3xy - 2y^2 = 27.$

24. $x^2 + 2xy + y^2 = 81,$
$x^2 - 2xy + y^2 = 9.$

25. $x^2 - 9y^2 = 16,$
$x^2 - y^2 = 24.$

26. $x^2(x - y) = 4,$
$x^2(2x + 3y) = 28.$

27. $xy^2 + xy = 24,$
$xy^2 + x = 40.$

28. $x^2 + y^2 = 13,$
$2xy - x - y = 7.$

29. $\dfrac{x + y}{x - y} = \dfrac{7}{3},$
$x - y^2 = 1.$

30. $\dfrac{x + y}{y} - \dfrac{2x}{x + y} + \dfrac{x^3 - x^2y}{y^3 - x^2y} = \dfrac{3}{5}.$
$y^2 - x^2 = 5.$

PROBLEMS.

1. Divide a into two parts, whose product shall equal b.

Let x and y represent the parts. Then, by the conditions,

$$x + y = a \qquad (1)$$

$$xy = b \qquad (2)$$

From (1), $$y = a - x$$

Substituting in (2), $$(a - x)x = b$$

$$x^2 - ax + b = 0$$

$$x = \tfrac{1}{2}a \pm \sqrt{\frac{a^2}{4} - b}$$

The two parts are $$\tfrac{1}{2}a + \sqrt{\frac{a^2}{4} - b}$$

and $$\tfrac{1}{2}a - \sqrt{\frac{a^2}{4} - b},$$

from which it appears that if $b > \frac{a^2}{4}$, the values are imaginary. Hence,

COR.—*The product of two quantities cannot be greater than the square of half their sum.*

Or, *The product of the two parts of a given quantity is greatest when those parts are equal.*

2. Find three numbers, the difference of whose differences is 8, their sum is 41, and the sum of their squares 699.

3. Find three numbers, the difference of whose differences is 5, their sum is 44, and their continued product is 1950.

4. The fore-wheel of a carriage makes 6 revolutions more than the hind-wheel in going 120 yards, but if the circumference of each wheel be increased one yard, it will make 4 revolutions more than the hind-wheel in the same distance. What is the circumference of each wheel?

5. What number being divided by the product of its two digits gives the quotient 2, and if 27 be added to the number, the digits will be inverted?

6. The difference between the hypothenuse and base of a right-angled triangle is 6, and the difference between the hypothenuse and perpendicular is 3. What are the sides?

7. A and B put out at interest different sums amounting to \$200. B's rate of interest was 1% more than A's. At the end of 5 years, B's accumulated simple interest was \$4 less than the double of A's. At the end of 10 years, A's principal and interest was $\frac{5}{8}$ of B's. What was each sum and rate per cent?

8. Two partners, A and B, gained \$140 in trade. A's money was 3 months in trade, and his gain was \$60 less than his stock, and B's money, which was \$50 more than A's, was in trade 5 months. What was each man's stock?

9. Find two numbers, the difference of whose squares is m^2, and which being multiplied respectively by a and b, the difference of the products is n^2.

10. Divide a and b each into two parts, such that the product of one part of a by one part of b shall be m, and the product of the remaining parts n.

11. What is the side of a cube which contains as many units of volume as there are linear units in its diagonal?

12. Find two numbers whose sum, product, and sum of their squares shall be equal to each other.

13. Find two numbers whose sum, product, and difference of their squares are equal to each other.

14. Find two numbers whose product equals the difference of their squares, and the sum of their squares equals the difference of their cubes.

15. The product of the sum and difference of two numbers is 8, and the product of the sum of their squares and the difference of their squares is 80. What are the numbers?

16. A and B bought 600 acres of land for \$600, each paying \$300. In dividing, A took the best land and paid 75 cents an acre more than B. How much land did each get and at what price?

CHAPTER XII.

INEQUATIONS.

351. An ***Inequation*** is an expression of *inequality* between two quantities; as, $a > b$ (read, "a is greater than b"); $x < y$ (read, "x is less than y").

The quantity on the left of the sign is called the ***First Member***, and the one on the right the ***Second Member*** of the inequation.

352. Algebraically, a negative quantity is said to be *less than zero* (Art. 104); and of two negative quantities, that which is *numerically greater* is *algebraically less*. Therefore,

If	$a - b > 0$,	$a > b$,
and if	$a - b < 0$,	$a < b$.

353. In the *transformation of inequations*, it is necessary to observe when the sign of inequality will be reversed.

When this sign is reversed, the *tendency* of the inequation is said to be changed.

Thus, $a > b$ and $c > d$, are inequations of the *same tendency*, and and $a > b$ and $c < d$ are of *opposite tendency*.

354. The tendency of an inequation is *not changed*,

1. *By any like operation upon both members, except changing their signs.*

2. *By adding or multiplying by the corresponding members of an inequation of the same tendency: Provided that in multiplying, the signs of the members be not changed.*

3. *By subtracting or dividing by the corresponding members of an inequation of opposite tendency: Provided that in dividing, the signs of the members be not changed.*

355. The tendency of an inequation *is changed,*

By changing the signs of both members. (Art. 104.)

For, $2 < 3$, but $-2 > -3$.

356. The tendency of an inequation *becomes doubtful,*

1. *By subtracting or dividing by the corresponding members of an inequation of the same tendency.*

2. *By adding or multiplying by the corresponding members of an inequation of opposite tendency.*

For it is evident that if $a < b$ and $c < d$,

$$a - c \lesseqgtr b - d,$$

$$\text{and} \quad \frac{a}{c} \lesseqgtr \frac{b}{d}.$$

Thus,	(1.)	$5 < 15$	and	$3 < 6$.
	Subtracting,		$2 < 9$.	
	Dividing,		$\frac{5}{3} < \frac{5}{2}$.	
	(2.)	$5 < 15$	and	$2 < 12$.
	Subtracting,		$3 = 3$.	
	(3.)	$5 < 15$	and	$1 < 3$.
	Dividing,		$5 = 5$.	
	(4.)	$5 < 6$	and	$1 < 4$.
	Dividing,		$5 > \frac{3}{2}$.	
	Subtracting,		$4 > 2$.	

357. The ***Reduction of an Inequation*** consists in so transforming it that the unknown quantity may stand alone as one member, while the other member contains only known quantities, the value of which is a limit to the value of the unknown quantity in one direction. If two inequations containing the same unknown quantity can be reduced with opposite tendencies, limits in both directions will be found.

EXAMPLES.

1. Given $\frac{x}{4} + 3 > \frac{x}{6} + 2$, to find a limit for x.

SOLUTION.—Clearing of fractions, $6x + 72 > 4x + 48$.
Transposing, $2x > -24$.
Dividing by 2, $x > -12$, *Ans.*

2. Given $\left\{ \begin{array}{l} 4x - 6 < 2x + 4 \\ 2x + 4 > 16 - 2x \end{array} \right\}$ to find limits for x.

SOLUTION.—Transposing and uniting terms,

$$2x < 10 \quad \text{and} \quad 4x > 12.$$

$$\therefore \quad x < 5 \quad \text{and} \quad x > 3, \quad \textit{Ans.}$$

3. Given $3x + 7x - 30 > 10$, to find a limit for x.

4. Given $x + \frac{1}{3}x - \frac{1}{2}x > 4$, to find a limit for x.

5. Given $\left\{ \begin{array}{l} \frac{ax}{5} + bx - ab > \frac{a^2}{5} \\ \frac{bx}{7} - ax + ab < \frac{b^2}{7} \end{array} \right\}$ to find limits for x.

6. Given $\left\{ \begin{array}{l} 3x - 4 < x + 6 \\ 5x + 7 > 3x + 13 \end{array} \right\}$ to find an integral value of x.

7. Given $\left\{ \begin{array}{l} \frac{1}{4}(x+2) + \frac{1}{3}x < \frac{1}{2}(x-4) + 3 \\ \frac{1}{4}(x+2) + \frac{1}{3}x > \frac{1}{2}(x+1) + \frac{1}{3} \end{array} \right\}$ to find an integral value of x.

8. A certain integral number, doubled and diminished by 7, is greater than 29; and 3 times the number diminished by 5 is less than double the number increased by 16. What is the number?

9. A boy sold a number of apples, such that triple the number increased by 2 exceeds double the number increased by 61; and 5 times the number diminished by 70 is less than 4 times the number diminished by 9. How many did he sell?

10. The sum of two whole numbers is 25. If the greater be divided by the less, the quotient will be greater than $\frac{7}{2}$; and if the less be divided by the greater, the quotient will be greater than $\frac{1}{3}$. What are the numbers?

CHAPTER XIII.

RATIO AND PROPORTION.

358. A ***Ratio*** is the *quotient* arising from the division of one quantity by another.

The sign of division commonly used to express ratio is the *colon* (:), as $a : b$, which means the ratio of a to b. But $a \div b$ and $\frac{a}{b}$ express the same thing.

359. A ***Proportion*** is an *equality of ratios*, or an equation each of whose members is a ratio; as $\frac{a}{b} = \frac{c}{d}$, or $a : b = c : d$.

NOTES.—1. The *double colon* (: :) is frequently used as the sign of equality in a proportion, but without good reason.

2. It is also questionable whether it were not better to express *ratio* in the *form of a fraction*, instead of using a *special sign* of *division*.

360. The ***First Term*** of a ratio is called the ***Antecedent;*** the ***Second Term*** the ***Consequent.***

361. The ***First*** and ***Last Terms*** of a proportion are called the ***Extremes;*** the ***Second*** and ***Third*** the ***Means.***

362. When the *same quantity* is used for *both means*, it is called a ***Mean Proportional*** between the other two; and the last term a ***Third Proportional*** to the other two. In this case the three quantities are said to be in ***Continued Proportion;*** as, $a : b = b : c$.

363. A ***Compound Ratio*** is the product of two or more ratios; as, $\frac{a}{b} \times \frac{c}{d} \times \frac{m}{n}$.

364. A ***Compound Proportion*** is one in which there is a compound ratio.

Theorem I.

365. *In any proportion, the product of the extremes equals the product of the means.*

Demonstration.—Let $a : b = c : d,$

Or $\frac{a}{b} = \frac{c}{d}$

Clearing of fractions, $ad = bc.$

Cor. 1.—*If three terms are in continued proportion, the square of the mean is equal to the product of the extremes.*

For, if $a : b = b : c, \quad \therefore \quad b^2 = ac.$

Cor. 2.—*A mean proportional between two quantities is the square root of their product.*

For, if $b^2 = ac, \quad \therefore \quad b = \sqrt{ac}.$

Cor. 3.—*The product of the extremes divided by one mean equals the other mean, and the product of the means divided by one extreme equals the other extreme.*

For, if $ad = bc, \quad \therefore \quad a = \frac{bc}{d},$ and $d = \frac{bc}{a}.$

Also $b = \frac{ad}{c},$ and $c = \frac{ad}{b}.$

Theorem II.

366. *If the product of two quantities be equal to the product of two other quantities, the factors of either product may be made the means with the factors of the other product for the extremes of a proportion.*

Dem.—Let $ad = bc;$

Dividing by bd, $\frac{a}{b} = \frac{c}{d};$

Or, $a : b = c : d.$

COR. 1.—*If the factors of one product are the same, that factor is a mean proportional between the other two.*

COR. 2.—*The means or extremes may change places; for the order in which the factors are taken is not material.*

If $a : b = c : d,$

Then $a : c = b : d.$

This change is called ***Alternation.***

THEOREM III.

367. *A proportion will remain true if both of its ratios be inverted.*

DEM.—If two quantities are equal, their reciprocals will be equal.

That is, if $\frac{a}{b} = \frac{c}{d},$

Then $\frac{b}{a} = \frac{d}{c}.$

This is called proportion by ***Inversion.***

THEOREM IV.

368. *The sum or difference of the terms of the first ratio is to either the antecedent or consequent of that ratio, as the sum or difference of the terms of the second ratio is to the antecedent or consequent of the second ratio.*

DEM.—Let $a : b = c : d,$

Or, $\frac{a}{b} = \frac{c}{d}.$

Then $\frac{a}{b} \pm 1 = \frac{c}{d} \pm 1,$

Or, $\frac{a \pm b}{b} = \frac{c \pm d}{d}.$

Again, $\frac{b}{a} = \frac{d}{c};$

$$1 \pm \frac{b}{a} = 1 \pm \frac{d}{c}.$$

$$\therefore \quad \frac{a \pm b}{a} = \frac{c \pm d}{c}.$$

Hence, $a \pm b : b = c \pm d : d;$

And $a \pm b : a = c \pm d : c.$

This is called proportion by ***Composition*** or ***Division;*** the former when the sum, the latter when the difference is used.

COR. 1.—*If four quantities are in proportion, the sum of the first and second is to the sum of the third and fourth as the difference of the first and second is to the difference of the third and fourth.*

For by Alternation,	$a + b : c + d = a : c,$
And	$a - b : c - d = a : c.$
But	$a : c = b : d;$
Therefore,	$a + b : c + d = a - b : c - d;$
Also	$a + b : a - b = c + d : c - d$; hence

COR. 2.—*The sum of the first and second is to their difference as the sum of the third and fourth is to their difference.*

COR. 3.—*In any number of equal ratios, the sum of the antecedents is to the sum of the consequents as any one antecedent is to its consequent.*

For if $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = r;$

$$\therefore \quad \frac{a + c + e + \&c.}{b + d + f + \&c.} = r = \frac{a}{b} = \&c. \quad \text{(Art. 200.)}$$

369. These changes may all be expressed as follows:

If four quantities are in proportion, they will be in proportion by ***Alternation, Inversion, Composition,*** or ***Division.***

THEOREM V.

370. *If four quantities are in proportion, the proportion will be true if both antecedents, both consequents, both terms of either ratio, or all the terms be multiplied by the same quantity.*

Let the student prove each of these by the properties of ratios.

THEOREM VI.

371. *If both antecedents or both consequents be increased or diminished by adding or subtracting quantities having the same ratio as the antecedents or consequents, the results will be in proportion with the antecedents or consequents, or with each other.*

DEM.—Let $a : b = c : d;$

Then $\frac{a}{b} \pm m = \frac{c}{d} \pm m;$

Or $\frac{a \pm mb}{b} = \frac{c \pm md}{d};$

Also $\frac{b}{a} \pm n = \frac{d}{c} \pm n;$

Or $\frac{b \pm na}{a} = \frac{d \pm nc}{c};$

$\therefore \quad a \pm mb : c \pm md = b \pm na : d \pm nc.$

THEOREM VII.

372. *A proportion is not destroyed by affecting each term by the same integral or fractional exponent.*

Let the student furnish the proof.

NOTE.—The ratio of the *squares* of two quantities is called the ***Duplicate Ratio,*** and the ratio of the *cubes* the ***Triplicate Ratio.***

Also the ratio of the *square roots* is called the ***Sub-duplicate,*** and of *cube roots* the ***Sub-triplicate*** ratio.

Theorem VIII.

373. *The products of the corresponding terms of any number of proportions are proportional.*

This is only the multiplication of several equations, member by member,

$$\text{As,} \qquad \frac{a}{b} = \frac{c}{d};$$

$$\frac{m}{n} = \frac{x}{y};$$

$$\therefore \qquad \frac{am}{bn} = \frac{cx}{dy}.$$

374. When two quantities have the same ratio as the reciprocals of two other quantities, they are said to be ***Reciprocally Proportional;*** as,

$$a : b = \frac{1}{c} : \frac{1}{d}.$$

This may be written $a : b = d : c$, in which the terms of the second ratio are inverted; hence they are also said to be *inversely proportional.*

This has no meaning unless a, b, c, and d are so related that c in some way belongs to a, and d to b. Otherwise the order $d : c$ would be no more an inverted order for the second ratio than $c : d$.

ILLUSTRATION.

Suppose two men travel at different rates the same distance. Then the times of travelling would be different.

Let r and r' be the rates, t and t' the corresponding times. We shall then have

$$r : r' = \frac{1}{t} : \frac{1}{t'} = t' : t.$$

The terms of the second ratio must here be taken in *inverse order*, or the *reciprocals* must be used. If they travel at the same rate but different times and distances, we have, putting d and d' for distances,

$$t : t' = d : d'.$$

In this case the quantities are said to be ***Directly Proportional;*** in the former they are ***Reciprocally*** or ***Inversely Proportional.***

PROBLEMS.

1. Find a third proportional to 25 and 50.

2. The last three terms of a proportion are 36, 24, and 16. What is the first term?

3. Find a mean proportional between 5 and 45?

4. If a men working b hours a day can complete a certain work in c days, in how many days can a' men working b' hours a day do it?

5. Two wagons with their loads have their weights in the ratio of 4 to 5. Parts of their loads in the proportion of 6 to 7 being removed, they weigh in the ratio of 2 to 3, and the sum of their weights is then 10 tons. What were the weights at first?

6. A starts to travel from C to D, and 3 hours afterwards B starts from D towards C, travelling 2 miles an hour more than A. When they meet the distances they have travelled are in the ratio of 13 to 15. If A had travelled 5 hours less and B had gone 2 miles an hour faster, they would have been in the ratio of 2 to 3. How many miles did each go, and how long did each travel before they met?

7. Find 3 numbers such that if 6 be added to the first and second the sums will be in the ratio 2 to 3, and if 5 be added to the first and third the sums will be in the ratio of 7 to 11, but if 36 be subtracted from the second and third, the remainders will be as 6 to 7.

8. Find the two numbers whose sum is to the less as 5 to 2, and whose difference multiplied by the difference of their squares is 135.

9. A certain number has 3 digits. The first is to the third as 16 to 6, the third to the second as 1 to 2, and the sum of the digits is 17. What is the number?

10. Find three numbers whose sum is to the first as 6 to 1, to the second as 3 to 1, and to the third as 2 to 1.

11. Find two numbers the ratio of whose difference to their sum is m, and the ratio of the difference of their squares to the sum of their squares is n.

VARIATION.

375. The ***Relations of Quantities*** are sometimes expressed by saying that *one varies* ***directly*** or ***inversely*** *as the other,* or as the *square* or *cube,* or some *other function* of the other.

376. The ***Sign of Variation*** ($\propto$) is used to express these relations ; as, $x \propto y$, which is read "x varies as y," and means that x and y are such functions of each other that their *ratio* remains the *same,* and that whatever changes may take place in the quantities themselves, the *increase* or *decrease* of one must always be *proportional* to the *increase* or *decrease* of the other.

377. The same thing may be expressed in other ways.

Let m represent a constant quantity ; that is, a quantity which in the same discussion does not change its value, and we may write

$$x \propto y; \quad x = my; \quad \text{or} \quad \frac{x}{y} = m,$$

each of which expresses the same relation.

378. If x_1 and x_2 represent two different values of x, and y_1 and y_2 the corresponding values of y, then the same thing is also expressed by the proportion

$$x_1 : x_2 = y_1 : y_2,$$

or

$$\frac{x_1}{x_2} = \frac{y_1}{y_2}.$$

379. If $x \propto \frac{1}{y}$ then

$$x = m \times \frac{1}{y} = \frac{m}{y},$$

or

$$x \div \frac{1}{y} = m,$$

or

$$xy = m.$$

In this case x varies ***reciprocally*** as y, while in $x \propto y$, x is said to vary ***directly*** as y.

380. From $x \propto \frac{1}{y}$ we have the proportion

$$x_1 : x_2 = \frac{1}{y_1} : \frac{1}{y_2};$$

in which x and y are said to be reciprocally proportional, or

$$x_1 : x_2 = y_2 : y_1;$$

in which they are inversely proportional. (Art. 374.)

EXERCISES.

381. Form the equations and proportions which are implied in the following:

1. $x \propto y^2$.

$x = my^2$; $\frac{x}{y^2} = m$; $x_1 : x_2 = y_1^2 : y_2^2$, *Ans.*

2. $x \propto \frac{1}{y^2}$.

3. $x \propto a + y$.

4. $x \propto y^2 + y^3$.

5. $x \propto \frac{1}{y + y^2}$.

6. If $x \propto y$, and $x = 3$ when $y = 5$, what is the value of y when $x = 15$? When $x = 7$? What of x when $y = 20$? When $y = 8$?

7. If $x \propto y^2$, and $x = 1$ when $y = 5$, what is the value of x when $y = 7$? 3? 2? 0?

8. If $z \propto mx + y$, and $z = 3$ when $x = 1$ and $y = 2$, and $z = 5$ when $x = 2$ and $y = 3$, what is the value of m?

9. If $x^2 \propto y^3$, and $x = 2$ when $y = 3$, what is the value of y in terms of x?

10. If $y \propto v + w$, $v \propto x$, and $w \propto \frac{1}{x}$; and, if $y = 4$ when $x = 1$, and $y = 5$ when $x = 2$, what is the value of y in terms of x.

11. If a body falls 192 inches the first second and the distance it falls varies as the square of the time, how far will it fall in 10 seconds? In how many seconds will it fall 400 ft.?

CHAPTER XIV.

PERMUTATIONS AND COMBINATIONS.

382. ***Permutations*** are the *different orders* in which things can be placed; as, *ab* and *ba*.

NOTE.—Observe that "*things*" does not mean quantities. Letters or figures which represent quantities may be the things whose permutations are considered, but they are regarded merely as so many objects which may be arranged in different orders.

383. ***Combinations*** are the *different groups* that can be formed of any number of things, taking a *given number* at a time.

Thus, from the letters *abc* we can form three groups if we take two at a time; viz., *ab*, *ac*, and *bc*. The *order* in which the individuals are placed in the group is not considered, *ab* and *ba* being but one combination.

384. The object of the *theory of permutations* and *combinations* is to determine the *number* of *orders* in which things can be placed, and the *number of groups* that can be formed.

385. The two problems may be thus stated:

I. To find the number of permutations of n things taken m at a time.

II. To find the number of combinations of n things taken m at a time.

NOTE.—For convenience we adopt the notation, P for permutations and C for combinations, and to indicate the number of things and the number taken at a time, write a subscript fraction whose *denominator* shall be the whole number of things and the *numerator* the number

taken at a time. Thus, $P_{\frac{3}{8}}$ = number of *permutations* of *8* things taken *3* at a time; and $C_{\frac{3}{8}}$ = number of *combinations* of *8* things taken *3* at a time.

386. To find a general expression for $P_{\frac{m}{n}}$ and $C_{\frac{m}{n}}$ we have, evidently, $P_{\frac{1}{n}} = n$, and if we take 2 at a time $P_{\frac{2}{n}} = n(n-1)$; for each of the n permutations formed by taking one at a time, may be placed before each of the $(n - 1)$ remaining things.

By the same reasoning we shall have

$$P_{\frac{3}{n}} = n\,(n-1)\,(n-2), \qquad \text{and}$$

$$P_{\frac{m}{n}} = n\,(n-1)\,(n-2)\,\ldots\,(n-m+1), \qquad (1)$$

$$P_{\frac{n}{n}} = n\,(n-1)\,(n-2)\,\ldots\,1 = \underline{|n}. \qquad (2)$$

387. Every combination of n things, taken m at a time, may have $P_{\frac{m}{m}} = \underline{|m}$ permutations.

Therefore, $$\underline{|m} \times C_{\frac{m}{n}} = P_{\frac{m}{n}}$$

or $$C_{\frac{m}{n}} = \frac{P_{\frac{m}{n}}}{\underline{|m}} = \frac{n\,(n-1)\,\ldots\,.\,(n-m+1)}{1\cdot 2\cdot 3\,\ldots\,.\,m}. \qquad (3)$$

388. If the things whose permutations are to be found are not all dissimilar, the number of permutations will be less, being evidently divided by the number of permutations which could be made with the identical things if they were unlike. That is, if p of the things are alike and q others are alike, etc., then

$$P_{\frac{n}{n}} = \frac{\underline{|n}}{\underline{|p} \times \underline{|q}, \text{ etc.}} \qquad (4)$$

EXAMPLES.

1. How many permutations can be made with the 9 digits taken all at a time? That is, $P_{\frac{9}{9}} =$ what?

2. $P_{\frac{5}{9}} =$ what?

3. $C_{\frac{5}{9}} =$ what?

4. $C_{\frac{4}{9}} =$ what?

5. Show that $C_{\frac{m}{n}} = C_{\frac{n-m}{n}}$.

6. How many products can be formed of six factors taken two at a time?

7. What is the number of products that can be formed from the 9 digits, taken 3 at a time?

8. How many more products can be formed of 50 numbers taken 40 at a time, than taken 10 at a time?

9. $P_{\frac{m}{n}} \div C_{\frac{m}{n}} =$ what?

10. $C_{\frac{8}{12}} - C_{\frac{9}{12}} =$ what?

11. If $P_{\frac{5}{n}} = 120 C_{\frac{3}{n}}$, what is the value of n?

12. Find the value of m that will make $C_{\frac{m}{2n}}$ the greatest possible.

13. Find the value of m that will make $C_{\frac{m}{2n+1}}$ the greatest possible.

14. How many permutations ($P_{\frac{n}{n}}$) can be made with the letters of the word *Permutations?* *Ans.* $\dfrac{\lfloor 12}{\lfloor 2}$.

15. How many permutations can be made with the letters of the word *Ecclesiastical?*

16. How many permutations can be made with the letters of the word *Divisibility?*

CHAPTER XV.

INFINITESIMAL ANALYSIS.

389. In the preceding chapters, quantities have been distinguished as *known* and *unknown*. The problems considered have involved quantities to which *arbitrary values could be assigned*, and others whose *values were to be found* from these. In these problems, changes in the values of quantities have been made by adding or subtracting *finite differences*.

390. But there is a large class of problems, in which quantities must be conceived as passing from one value to another by a *process of growth*.

A quantity changing in this way from one value to another, *passes through all intermediate values;* as, when a point moves from one position to another, it passes through all intermediate positions; or as time in passing from one hour to another, passes through every instant of intermediate time.

This leads to the conception of quantities as *constant* and *variable*.

391. A ***Variable*** is a quantity conceived as changing from one value to another in such manner as to pass through all intermediate values.

392. A ***Constant*** is a quantity whose value remains the same during the same discussion.

Constants are of two kinds, *absolute* and *arbitrary*.

393. An ***Absolute Constant*** is a quantity expressed by a number whose value *never* changes; as, 4, 10, etc.

394. An ***Arbitrary Constant*** is represented by one or more letters, to which values may be *arbitrarily* assigned, but which remain the same throughout the same discussion.

395. ***Consecutive Values*** of a variable are those values between which there are no intermediate values.

396. The ***Differential*** of a *variable* is the *difference* between two of its consecutive values.

NOTE.—It is evidently impossible to conceive of this difference; for any conceivable difference would imply intermediate values; but the existence of *strictly consecutive values* is a *necessity* from the manner in which a variable changes its value.

397. A ***Function*** (in the infinitesimal analysis) is a quantity or algebraic expression whose value depends on one or more variables.

A *function* is therefore a *variable quantity*, having its *consecutive values* corresponding to the consecutive values of the variables on which it depends.

398. The ***Differential*** of a *function* is the *difference* between two of its consecutive values.

399. We may suppose the *infinitesimal increments* or *differentials* by which a variable passes from one value to another to be equal to each other; that is, we may assume the *differential of a variable* to be *constant*, since there is nothing to forbid such a supposition, but the *differential of the function* which depends on the differential of the variable will *not usually* be constant. Hence,

400. The ***Differential of a Function*** may be defined as *the infinitesimal change in the function produced by a change in the variable from one value to its consecutive value.*

401. ***Differentiation*** is the process of finding a *general expression* for the difference between any two consecutive values of a function; in other words, of finding a *general expression* for the *differential of a function.*

402. The variable whose increments are arbitrarily assumed is called the ***Independent Variable.*** (Art. 399.)

403. The function whose increments depend on the increments of the independent variable is called the ***Dependent Variable.***

Thus, in the expression $x^2 - x$, if we assume that x increases by a constant increment, x will be the *independent* and $x^2 - x$ the *dependent* variable, or, making $u = x^2 - x$, u is the *dependent variable.*

404. *Infinitesimal quantities,* expressed in terms of any finite unit, are all o. (Art. 42, 6th.) In order, therefore, to express the relations of infinitesimals to each other, some infinitesimal unit must be employed. It need be no objection to the use of such a unit, that the conception of its magnitude is impossible, for mathematics is concerned only with the *ratios of quantities*, and never with their *absolute magnitude.* It is of no advantage whatever to the mathematician to know the magnitude of the unit employed.

We may therefore assume as the *unit of measure* for differentials the ***Differential of the Independent Variable.*** (Art. 399.)

405. To ***Differentiate a Function*** will then be, to find the *differential of the function* in terms of the *differential of the variable* on which the function depends.

NOTATION.

406. The ***Differential of a Variable*** is expressed by writing before it the letter d; thus, dx, dy, dz, etc., to be read, "differential of x," "differential of y," etc.

NOTE.—The d in these expressions is *not a factor*, but only an *abbreviation* of the word *differential*, and must be so read.

The same abbreviation placed before a function indicates the differential of the function; as, $d(x^2)$, $d(x^2 - x)$, are read and mean, "the differential of x^2," "the differential of $x^2 - x$."

In all such cases, the function whose differential is expressed must be enclosed in a parenthesis. Thus, $d(x^2)$ is the differential of x^2; but dx^2 is the square of the differential of x.

407. When any function of a variable is under discussion, to avoid its repetition, we write the variable in parentheses, with f, ϕ, or ψ before it; as, $f(x)$, $f(y)$, etc., read, "function of x," "function of y," etc.

Different functions are expressed by $f'(x)$, $f_1(x)$, $\phi(x)$, etc., read, "function prime of x," "function sub one of x," "the ϕ function of x," etc.

408. When either of these symbols is used for any function, it represents the same function throughout the same discussion.

409. The differential of a function is usually a *variable*, and may therefore be differentiated. Its differential is called a *second* differential, and is expressed thus, d^2u, d^2y, read, "second differential of u," etc. The meaning of these will be more fully explained hereafter.

410. The ***Differential Coefficient*** of a *function* is the *ratio* of the *differential of the function* to the *differential of the variable* on which the function depends. If u represent the function and x the variable, the differential coefficient is expressed by $\frac{du}{dx}$.

411. Infinitesimals and infinites are of *different orders*, depending on the *number* of such factors they contain.

Thus, dx, dy, dz, etc., are of the first order, having but one infinitesimal factor; dx^2, dy^2, $dx\,dy$, etc., are of the second order, and dx^3, dy^3, $dx^2\,dy$, $dx\,dy\,dz$, etc., are of the third order, and so on for higher orders.

So also ∞^2 is of the second, and ∞^3 of the third order.

It is evident that an infinitesimal of the *second* order is infinitely less than one of the *first*; one of the *third* infinitely less than one of the *second*, and so on; for each additional infinitesimal factor divides the quantity by infinity or multiplies it by $\frac{1}{\infty}$.

In like manner, ∞^2 is infinitely greater than ∞.

8

412. We have seen that when a quantity is measured with a unit infinitely greater than itself, as when an infinitesimal is measured with a finite unit, the measure is zero. (Art. 404.) Hence it follows that,

In any polynomial, a term that is infinitely less than another term may be treated as o *and omitted.*

That is, in an expression containing finite and infinitesimal terms, the infinitesimal terms may be dropped; and in expressions containing only infinitesimal terms, all higher orders of infinitesimals may be dropped. Thus,

$$a \pm dx = a;$$
$$dx \pm dx^2 = dx;$$
$$dx \pm dx\,dy = dx.$$

So also, $\infty \pm a = \infty$,

and transposing, $\infty - \infty = \pm a$. (Art. 348.)

DIFFERENTIATION.

413. By the definition of a *differential* (Art. 400), we may evidently ***Differentiate a Function*** by the following

GENERAL RULE.

I. *Add to the variable the differential of the variable.*

II. *Develop and simplify the expression.*

III. *Subtract the primitive state of the function.*

414. The application of this rule to different functions leads to *special rules* by which the process of differentiation is *abbreviated.*

1. Differentiate ax.

Let	$u = ax.$	(1)
Adding dx to x,	$u + du = a(x + dx).$	
Developing,	$u + du = ax + adx.$	
Subtracting (1),	$du = adx.$	Hence,

Rule I.—*The differential of the product of a variable by a constant factor is the product of the constant factor by the differential of the variable.*

2. Differentiate $ax - bx + c$.

Let $u = ax - bx + c$. (1)

Adding dx to x, $u + du = a(x + dx) - b(x + dx) + c$.

Developing, $u + du = ax + adx - bx - bdx + c$.

Subtracting (1), $du = adx - bdx$. Hence,

RULE II.—*The differential of a polynomial is the algebraic sum of the differentials of its several terms, the differential of a constant term being zero.*

3. Differentiate vyz, in which v, y, and z represent functions of x.

Let $u = vyz$. (1)

Adding dx to x, $u + du = (v + dv)(y + dy)(z + dz)$.

Developing, etc., $u + du = vyz + vydz + vzdy + yzdv$.

Subtracting (1), $du = vydz + vzdy + yzdv$. Hence,

RULE III.—*The differential of the product of several functions is the sum of the products of the differential of each function by the product of the other functions.*

NOTES.—1. Observe that as v, y, and z, are functions of x, adding dx to x adds to each function the differential of that function, that is, dv to v, dy to y, and dz to z.

2. In reducing the expression after it is developed, the terms $v\,dy\,dz + y\,dv\,dz + z\,dv\,dy + dv\,dy\,dz$ are dropped, being infinitesimals of the second and third orders.

3. It may be shown that this rule applies to any number of factors.

4. Differentiate $\frac{y}{z}$, y and z being functions of x.

Let $u = \frac{y}{z}$. (1)

Adding dx to x, $u + du = \frac{y + dy}{z + dz}$.

Subtracting (1), $du = \frac{y + dy}{z + dz} - \frac{y}{z}$.

Reducing, $du = \frac{zdy - ydz}{z^2}$. Hence,

RULE IV.—*The differential of a fraction is the denominator multiplied by the differential of the numerator minus the numerator multiplied by the differential of the denominator, divided by the square of the denominator.*

NOTES.—1. If the numerator be constant, its differential is zero and the expression becomes $\frac{-y\,dz}{z^2}$.

2. If the denominator be constant; as $\frac{y}{a}\left(=\frac{1}{a}y\right)$, it may be differentiated by Rule I.

5. Differentiate x^n.

1st. When n is a positive integer.

Let $\qquad u = x^n = x\,.\,x\,.\,x\ \ .\ \ .$ to n factors.

By Rule III, $\qquad du = x^{n-1}dx + x^{n-1}dx +$ &c., to n terms.

$\therefore \qquad du = nx^{n-1}dx.$

2d. When n is a positive fraction.

Let $\qquad n = \frac{m}{s}$, m and s being positive integers

Then $\qquad u = x^{\frac{m}{s}}$,

and $\qquad u^s = x^m.$

$\therefore \qquad su^{s-1}du = mx^{m-1}dx,$

and $\qquad du = \frac{mx^{m-1}}{su^{s-1}}dx.$

Substituting $x^{\frac{m}{s}}$ for u in the denominator,

$$du = \frac{mx^{m-1}}{s\left(x^{\frac{m}{s}}\right)^{s-1}}dx = \frac{mx^{m-1}}{sx^{m-\frac{m}{s}}}dx = \frac{m}{s}x^{\frac{m}{s}-1}dx.$$

3d. When n is negative, and either integral or fractional.

Let $\qquad u = x^{-n} = \frac{1}{x^n}.$

By Rule IV, $\qquad du = -\frac{nx^{n-1}dx}{x^{2n}} = -nx^{-n-1}dx.$

Hence, whatever be the value of n,

RULE V.—*The differential of any power of a variable whose exponent is constant, is the continued product of the exponent, the variable with its exponent diminished by one, and the differential of the variable.*

415. The formulas of which these rules are the translation are more convenient to memorize and use than the rules themselves. We therefore collect them below.

I. $d(ax) = adx.$

II. $d(ax - bx + c) = adx - bdx.$

III. $d(xyz) = xydz + xzdy + yzdx.$

IV. $d\left(\dfrac{x}{y}\right) = \dfrac{ydx - xdy}{y^2}.$

V. $d(x^n) = nx^{n-1}dx.$

EXERCISES.

1. Differentiate $5x^3 - 3x^2 + 7x - 6$. (See I, II, and V.)
Ans. $(15x^2 - 6x + 7)\,dx.$

2. Differentiate $(x - a)(x + b)$. (See III.)
Ans. $(x - a)\,dx + (x + b)\,dx = (2x - a + b)\,dx.$

3. Differentiate $\dfrac{a - x}{a + x}$. (See IV.)

$$\textit{Ans.}\quad \frac{-(a + x)\,dx - (a - x)\,dx}{(a + x)^2} = \frac{-2a}{(a + x)^2}\,dx.$$

4. Differentiate $y = ax^2 + bx - c$.
Ans. $dy = (2ax + b)\,dx.$

Dividing by dx, we have $\dfrac{dy}{dx} = 2ax + b.$

416. This is called the *first differential coefficient* of the function $ax^2 + bx - c$. (Art. 410.) It is the factor which multiplied by dx gives dy, or the differential of the function.

This function $(2ax + b)$ may be differentiated again, or we may differentiate

$$dy = (2ax + b)\,dx.$$

Remembering that dx is constant and differentiating, we have

$$d^2y = 2adx^2,$$

which is the second differential of the function

$$ax^2 + bx - c.$$

Dividing by dx^2, we have

$$\frac{d^2y}{dx^2} = 2a.$$

417. This is called the *second differential coefficient* of the function. It is the ratio of the second differential of the function to the square of the differential of the variable. This function ($2a$) is constant, therefore its differential will be zero, and the third differential of $ax^2 + bx - c$ is zero.

Differentiate the following and find their successive differential coefficients.

5. $y = ax^4 - bx^3 + cx - ab.$

Ans. $$\frac{dy}{dx} = 4ax^3 - 3bx^2 + c.$$

$$\frac{d^2y}{dx^2} = 12ax^2 - 6bx.$$

$$\frac{d^3y}{dx^3} = 24ax - 6b.$$

$$\frac{d^4y}{dx^4} = 24a.$$

$$\frac{d^5y}{ax^5} = 0.$$

6. $y = 9x^5 + 5x^4 + 3x^3 - 7x^2 - 5.$

Ans. $$\frac{dy}{dx} = 45x^4 + 20x^3 + 9x^2 - 14x.$$

$$\frac{d^2y}{dx^2} = 180x^3 + 60x^2 + 18x - 14.$$

$$\frac{d^3y}{dx^3} = 540x^2 + 120x + 18.$$

$$\frac{d^4y}{dx^4} = 1080x + 120.$$

$$\frac{d^5y}{dx^5} = 1080.$$

$$\frac{d^6y}{dx^6} = 0.$$

7. $y = (x - a)(x - b)(x - c).$

8. $y = \dfrac{x^2 + 1}{x + 1}.$

9. $y = \dfrac{x}{x^2 + 1}.$

10. $y = x^{\frac{3}{5}} - 9x^{\frac{4}{3}} - 5x^{\frac{4}{3}} + 7.$

11. $y = \dfrac{1}{1 - x}.$

12. $y = A + Bx + Cx^2 + Dx^3.$

13. $y = (1 + x)^5.$

Considering the binomial $1 + x$ the variable, we have

$$dy = 5(1 + x)^4\, d(1 + x) = 5(1 + x)^4\, dx. \quad \text{(Art. 415, V.)}$$

14. $y = (1 - x)^4.$

15. $y = (1 + x)^n.$

16. $y = (1 - x)^m.$

17. $y = x^7 - 5x^4 + 3x^2 + 5.$

18. $y = (x - 1)(x + 2)(x - 5)(x + 3).$

19. $y = \sqrt{\dfrac{a^2 - x^2}{1 + x}}.$

$\sqrt{\dfrac{a^2 - x^2}{1 + x}} = \dfrac{\sqrt{a^2 - x^2}}{\sqrt{1 + x}}$, the differential of which is

$$\frac{\sqrt{1 + x} \times d\sqrt{a^2 - x^2} - \sqrt{a^2 - x^2} \times d\sqrt{1 + x}}{1 + x}. \quad \text{(Art. 415, IV.)}$$

Therefore, reducing, $$dy = -\frac{x^2 + 2x + a^2}{2(a^2 - x^2)^{\frac{1}{2}}(1 + x)^{\frac{3}{2}}}\, dx.$$

20. $y = \sqrt{\dfrac{ax}{by}}.$

21. $y = (1 + x)^{-\frac{1}{2}}.$

22. $y = (1 - x^2)^{-2}.$

23. $y = \sqrt[3]{a + x^2}.$

24. $u = x^2y^3 + x^3y^2.$

25. $u = \frac{1}{3}x^3 - \frac{1}{2}x^2 + 5.$

CHAPTER XVI.

INDETERMINATE COEFFICIENTS.

418. ***Indeterminate Coefficients*** are letters assumed to represent certain *unknown coefficients* during the process by which these coefficients are determined.

419. The ***Theory of Indeterminate Coefficients*** is expressed by the following

THEOREM.

If a polynomial function of a single variable be equal to zero for all values of that variable, the coefficients of the different powers of the variable will each be equal to zero.

DEMONSTRATION.—Let

$$Ax^a + Bx^b + Cx^c + Dx^d + \text{etc.} = 0 \qquad (1)$$

be the given equation, in which $a < b < c$, etc., and a is positive. This involves no contradiction of the hypothesis, for the terms of the equation may be arranged in any order, and the equation may be cleared of fractions without affecting its truth. Dividing this equation by x^a gives

$$A + Bx^{b-a} + Cx^{c-a} + Dx^{d-a} + \text{etc.} = 0.$$

Since this equation is true for all values of x, it is true when $x = 0$, a supposition which gives $A = 0$. The term Ax^a may therefore be dropped, and the resulting equation divided by x^b, giving

$$B + Cx^{c-b} + Dx^{d-b} + \text{etc.} = 0.$$

Making $x = 0$ gives $B = 0$. In like manner, we may show that all the coefficients are zero.

COR.—*If an equation between two polynomial functions of a single variable be true for all values of that variable, the equation will be identical.*

For, transposing all the terms to one member, the coefficients will all become zero, which would not be the case if the equation were not identical; that is, the coefficients of the like powers of the variable will be the same in the two members.

420. The most important applications of the Theory of Indeterminate Coefficients are,

1. To the *development of functions.*
2. To the *decomposition of fractions.*
3. To the *reversion of series.*

421. A function is ***developed*** when some indicated operation is performed.

EXAMPLES.

1. Develop $\frac{1}{1+x}$.

OPERATION.

Assume $\frac{1}{1+x} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc.}$ (1)

Clearing of fractions and transposing the terms to the second member,

$$0 = \begin{array}{c|c|c|c|c|c|l} A & +A & x+B & x^2+C & x^3+D & x^4+E & x^5 + \text{etc.} \\ -1 & +B & +C & +D & +E & +F & \end{array} \quad (2)$$

Since x is a variable, equation (1) is true for all values of x, and we have from Equation (2), by the Theorem,

$$A - 1 = 0; \quad A + B = 0; \quad B + C = 0; \quad C + D = 0; \quad \text{etc.}$$

$$\therefore \quad A = 1; \quad B = -1; \quad C = 1; \quad D = -1; \quad \text{etc.}$$

NOTE.—The values of these coefficients enable us to determine the law of the series, and to write any required number of terms, as follows:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \text{etc.}$$

2. Develop $(a - x)^{\frac{1}{2}}$.

OPERATION.

Assume $(a - x)^{\frac{1}{2}} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc.}$ (1)

Squaring,

$$a - x = A^2 + 2ABx + \begin{array}{l|} 2AC \\ + B^2 \\ \\ \end{array}\, x^2 + \begin{array}{l|} 2AD \\ + 2BC \\ \\ \end{array}\, x^3 + \begin{array}{l|} 2AE \\ + 2BD \\ + C^2 \end{array}\, x^4 + \text{etc.}$$

$\therefore$ Art. 419, Cor.,

$$\begin{array}{rl} A^2 = a; & A = a^{\frac{1}{2}}; \\ 2AB = -1; & B = -\dfrac{1}{2a^{\frac{1}{2}}}; \\ 2AC + B^2 = 0; & C = -\dfrac{1}{8a^{\frac{3}{2}}}; \\ 2AD + 2BC = 0; & D = -\dfrac{1}{16a^{\frac{5}{2}}}; \\ 2AE + 2BD + C^2 = 0; & E = -\dfrac{5}{128a^{\frac{7}{2}}}; \\ \text{etc.} & \text{etc.} \end{array}$$

Substituting in (1),

$$(a - x)^{\frac{1}{2}} = a^{\frac{1}{2}} - \frac{x}{2a^{\frac{1}{2}}} - \frac{x^2}{8a^{\frac{3}{2}}} - \frac{x^3}{16a^{\frac{5}{2}}} - \frac{5x^4}{128a^{\frac{7}{2}}}, \quad \text{etc.}$$

422. From these solutions we have for the development of a function of a single variable the following

RULE.—I. *Assume the function equal to a series with indeterminate coefficients, containing all the powers of the variable which the development requires.*

II. *Free this equation from fractions and from parentheses which include different powers of the variable, and make the coefficients of like powers of the variable in the two members equal to each other; or transpose all the terms to one member, and make the several coefficients equal to zero.*

III. *From the equations thus formed find the values of the coefficients and substitute them in the assumed series.*

NOTES.—1. The form of the function must determine what powers of the variable to assume. One thing should always be observed, viz.: The assumed equation should give no absurd result when $x = 0$. Thus, if in equation (1), Example 1, $x = 0$, $A = 1$, a result involving no

absurdity; but if the series were $Ax + Bx^2 + Cx^3$, etc., $x = 0$ would give $1 = 0$, an absurdity.

2. If any power or powers of the variable belonging to the development are omitted from the assumed series, it will be shown by some such absurdity in the course of the solution, if it does not appear by making $x = 0$.

3. If powers of the variable not found in the development be assumed, their coefficients will be found to be zero, and the function will be correctly developed. It appears, therefore, that it is only necessary to make sure of including all necessary powers, since it does not vitiate the development to include those that are unnecessary.

Let the student illustrate this by assuming for the function above,

$$\frac{1}{1 + x} = A + Bx^2 + Cx^4 + Dx^6 + \text{etc.}$$

Also, $$\frac{1}{1 + x} = Ax^{-2} + Bx^{-1} + C + Dx + Ex^2 + \text{etc.}$$

Develop the following:

3. $\dfrac{x}{1 + x}$.

4. $\dfrac{1}{x + x^2}$.

5. $(x - 1)^{\frac{1}{2}}$.

6. $\dfrac{ax - x^2}{x^3 - x^4}$.

7. $(a + x)^{-\frac{1}{2}}$.

8. $\dfrac{a}{(x + a)^2}$.

9. $\dfrac{x^2 - 1}{x^3 + x^2 - 2x + 1}$.

10. $\dfrac{x + 1}{1 - x^2}$.

11. $\dfrac{(1 - x)^3}{1 - x^3}$.

12. $(a + x)^{-1}$.

13. $\dfrac{x^2 - 2x + 1}{x^3 - x^2}$.

14. $a\,(a - x)^{-1}$.

15. $(x - 1)^{\frac{1}{3}}$.

16. $\dfrac{1 - \sqrt{x}}{x - \sqrt{x}}$.

17. $\dfrac{x + \sqrt{x}}{x^2 - x^{\frac{3}{2}} + x^{\frac{1}{2}}}$.

18. $\dfrac{x - x^{\frac{1}{3}}}{(1 + x^{\frac{1}{3}})^3}$.

19. $\dfrac{1}{1 - x^{\frac{1}{4}}}$.

20. $\dfrac{x^{\frac{1}{2}}}{(1 + x^{\frac{1}{2}})(1 - x^{\frac{1}{2}})}$.

NOTE.—Observe what powers of x will be found in the development of the last five examples.

DECOMPOSITION OF FRACTIONS.

423. A ***Rational Fraction***, a function of a single variable, whose denominator has rational factors, may be separated into two or more fractions, whose sum shall be equal to the given fraction.

This is called *decomposing the fraction*, and the several fractions are called *partial fractions*.

424. The fraction to be decomposed is understood to have its *numerator* of a lower degree than its *denominator;* otherwise it would give by division one or more integral terms.

425. The ***Decomposition of a Fraction*** is performed by the following

RULE.—I. *Assume the given fraction equal to the sum of several fractions with indeterminate numerators, and whose denominators include all the denominators possible for the partial fractions.*

II. *Clear the equation of fractions, and collect like powers of the variable in one term.*

III. *Equate the coefficients of these like powers, and from the equations thus formed determine the values of the numerators.*

IV. *Substitute these values in the assumed fractions.*

426. The manner in which the numerators and denominators of the partial fractions are assumed needs special notice. By the rule, we are to include in the denominators all denominators possible to the partial fractions.

To ascertain what these will be, it should be observed that, since the given fraction is the *sum* of the *partial fractions*, each denominator must be a *factor* of the given denominator. (Art. 197.)

The *denominator* of the given fraction being a rational function of a single variable, the *prime rational factors* will have one of the following forms,

$$x, \quad x \pm a, \quad \text{or} \quad x^2 \pm ax + b. \quad \text{(Art. 549, Cor. 2.)}$$

Any one or more of these forms may be found with any exponent, so that

$$x^n, \quad (x \pm a)^n, \quad \text{and} \quad (x^2 \pm ax + b)^n,$$

will represent all the different factors of the given denominator.

427. Considering the first of these, x^n, we see that it must be one of the partial denominators, otherwise it would not be a factor of the denominator of the sum. (Art. 197.)

So also x^{n-1} *may be* one denominator, and in like manner x^{n-2}, x^{n-3}, ... x, are all *possible* denominators. Hence they must all be used, and from x^n we shall have the partial denominators x, x^2, x^3, ... x^n.

In like manner, $(x \pm a)^n$ will give the denominators

$$x \pm a, \quad (x \pm a)^2, \quad . \quad . \quad . \quad . \quad (x \pm a)^n,$$

and $(x^2 + ax + b)^n$ will give

$$x^2 \pm ax + b, \quad (x^2 \pm ax + b)^2, \quad . \quad . \quad . \quad . \quad (x^2 \pm ax + b)^n.$$

428. To determine what numerator to assume for each denominator, consider first the proper form of the numerator for x^n. This numerator must be independent of x, for if it should contain x, $\left(\text{as } \dfrac{Ax + B}{x^n}\right)$, it would form a fraction capable of further decomposition, and the partial fractions would not be the simplest possible. The assumed fraction therefore must be of the form $\dfrac{A}{x^n}$.

The same may be said of all the fractions with monomial denominators.

429. The numerator also for $(x - a)^n$ must be independent of x for the same reason, and therefore all the fractions having denominators of that form will have numerators of the zero degree,

430. The denominators of the form $(x^2 \pm ax + b)^n$ are quadratic factors, whose binomial factors are imaginary, and therefore cannot be further resolved. Their numerators may therefore contain the first as well as the zero power of x, and must be in the form $Ax + B$.

This will give for the different forms of the partial fractions,

$$\frac{A}{x^n}, \qquad \frac{B}{(x \pm a)^n}, \qquad \text{and} \qquad \frac{Cx + D}{(x^2 \pm ax + b)^n}.$$

In the third form, a may be zero, reducing it to

$$\frac{Cx + D}{(x^2 + b)^n}.$$

The process will be made plainer by the solution of a few

EXAMPLES.

1. Decompose $\dfrac{x^5 - 3x^3 + 1}{x^2 (x - 2)^3 (x^2 + 2)^2 (x^2 - 2x + 2)^2}$.

SOLUTION.—Assuming the partial fraction as above indicated,

$$\frac{x^5 - 3x^3 + 1}{x^2 (x-2)^3 (x^2+2)^2 (x^2-2x+2)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} + \frac{D}{(x-2)^2} + \frac{E}{(x-2)^3}$$

$$+ \frac{Fx+G}{x^2+2} + \frac{Hx+I}{(x^2+2)^2} + \frac{Kx+L}{x^2-2x+2} + \frac{Mx+N}{(x^2-2x+2)^2}.$$

NOTE.—This example is given for the purpose of including all possible forms of partial fractions, and the student should compare it with the preceding explanations. The complete solution would occupy too much space to be conveniently printed. The student may complete it for his own practice and satisfaction.

2. Decompose $\dfrac{x^2 - 2x + 2}{x^3 + 2x^2 - x - 2}$.

SOLUTION.—The factors of the denominator are $x+1$, $x-1$, and $x+2$. Assume

$$\frac{x^2 - 2x + 2}{x^3 + 2x^2 - x - 2} = \frac{A}{x + 1} + \frac{B}{x - 1} + \frac{C}{x + 2}.$$

Clearing of fractions and uniting terms,

$$x^2 - 2x + 2 = \begin{array}{r|r|r|l} -2A & +A & x + A & x^2 \\ +2B & +3B & +B & \\ -C & & +C & \end{array}$$

$$\therefore \quad -2A + 2B - C = 2.$$

$$A + 3B = -2.$$

$$A + B + C = 1.$$

From which, $A = -\frac{5}{2}$, $B = \frac{1}{6}$, $C = \frac{10}{3}$.

$$\therefore \quad \frac{x^2 - 2x + 2}{x^3 + 2x^2 - x - 2} = -\frac{5}{2(x+1)} + \frac{1}{6(x-1)} + \frac{10}{3(x+2)}.$$

3. Decompose $\dfrac{x^2 - x + 1}{x(x^2 + 3x - 10)}$.

4. Decompose $\dfrac{2x^2 - 3}{3x^3 + 6x^2 + 3x}$.

5. Decompose $\dfrac{x^3 - 1}{(x^2 - 2x + 1)(x^2 + x + 4)}$.

6. Decompose $\dfrac{x^4 + x^2 + 1}{x(x^2 + 2x + 7)^2}$.

7. Decompose $\dfrac{x^3 - 3x + 3}{x(x^2 - 2x - 15)}$.

8. Decompose $\dfrac{ax - a^2}{x^3 - 2x^2 + x}$.

9. Decompose $\dfrac{1}{x^4 - 1}$.

10. Decompose $\dfrac{1}{(x-2)^2(x+1)^2}$.

11. Decompose $\dfrac{1}{x^2 + ax - bx - ab}$.

12. Decompose $\dfrac{2x^5 + 3x^4 - 7x^3 + 9x^2 - 6x + 4}{(x^2 - 4)^2(x^2 - 1)}$.

CHAPTER XVII.

DEMONSTRATION OF THE BINOMIAL THEOREM.

431. The ***Binomial Formula*** has already been given, and the student is familiar with its use. It only remains to give the proof, which he was not prepared to understand at an earlier stage of his progress.

432. Let it be required to develop $(a + x)^n$ into a series of ascending powers of x, n being any number whatever, *positive* or *negative*, *integral* or *fractional*.

$$(a + x)^n = a^n\left(1 + \frac{x}{a}\right)^n.$$

Put $$\frac{x}{a} = z,$$

Then $$a^n\left(1 + \frac{x}{a}\right)^n = a^n(1 + z)^n,$$

and the development of $(1 + z)^n$ will give the required series when the value of z is restored and the series multiplied by a^n. Assume,

$$(1 + z)^n = A + Bz + Cz^2 + Dz^3 + Ez^4 + \text{etc.,} \qquad (1)$$

in which A, B, C, etc., are *indeterminate coefficients*, whose values are to be found.

Performing the successive differentiations of (1) and dividing each by dz, we have

$$n(1+z)^{n-1} = B + 2Cz + 3Dz^2 + 4Ez^3 + 5Fz^4 + \text{etc.} \qquad (2)$$

$$n(n-1)(1+z)^{n-2} = 2C + 2.3Dz + 3\cdot 4Ez^2 + 4\cdot 5Fz^3 + \text{etc.} \qquad (3)$$

$$n(n-1)(n-2)(1+z)^{n-3} = 2\cdot 3D + 2\cdot 3\cdot 4Ez + 3\cdot 4\cdot 5Fz^2 + \text{etc.} \qquad (4)$$

$$n(n-1)(n-2)(n-3)(1+z)^{n-4} = 2\cdot 3\cdot 4E + 2\cdot 3\cdot 4\cdot 5Fz + \text{etc.} \qquad (5)$$

$$n(n-1)(n-2)(n-3)(n-4)(1+z)^{n-5} = 2\cdot 3\cdot 4\cdot 5F + \text{etc.} \qquad (6)$$

Making $z = 0$, we have from these equations

$$A = 1; \qquad B = n;$$

$$C = \frac{n(n-1)}{1 \cdot 2};$$

$$D = \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3};$$

$$E = \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4};$$

$$F = \frac{n(n-1)(n-2)(n-3)(n-4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}, \text{ etc.};$$

from which we readily determine the *law* of the coefficients, and may write the series,

$$(1 + z)^n = 1 + nz + \frac{n(n-1)}{1 \cdot 2} z^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} z^3 + \text{etc.}$$

Restoring the value of z and multiplying by a^n, we have the formula as given in Art. 268,

$$(a + x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1 \cdot 2} a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^{n-3}x^3$$

$$+ \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} a^{n-4}x^4 + \text{etc.}$$

433. The m^{th} term of the series is

$$\frac{n(n-1)(n-2) \ldots\ldots (n-m+2)}{1 \cdot 2 \cdot 3 \cdot 4 \ldots\ldots (m-1)} a^{n-m+1}x^{m-1}.$$

The $(m + 1)^{th}$ term is

$$\frac{n(n-1)(n-2) \ldots\ldots (n-m+2)(n-m+1)}{1 \cdot 2 \cdot 3 \cdot 4 \ldots\ldots (m-1)m} a^{n-m}x^m.$$

Dividing the $(m + 1)^{th}$ term by the m^{th} term gives

$$\frac{n-m+1}{m} \cdot \frac{x}{a} = \left(\frac{n+1}{m} - 1\right)\frac{x}{a}.$$

This is the ***variable factor*** by which any term may be multiplied to produce the next, m representing the number of the term multiplied.

EXAMPLES.

1. Expand by the formula, $(a + b)^5$.
2. Expand by the formula, $(a - b)^{\frac{1}{2}}$.
3. Expand by the formula, $(a - b)^{-3}$.
4. Expand by the formula, $(a + b)^{-1}$.
5. Expand by the formula, $(x - y)^{-\frac{1}{2}}$.
6. Expand by the formula, $\dfrac{1}{(x - y)^2}$.
7. Expand by the formula, $(m + n)^{\frac{2}{3}}$.
8. Expand by the formula, $\dfrac{1}{(m - n)^{\frac{2}{3}}}$.
9. Expand by the formula, $\dfrac{1}{(m + n)^{\frac{2}{3}}}$.
10. Expand by the formula, $(x - 2)^{\frac{2}{3}}$.
11. Expand by the formula, $(x + 2)^3$.
12. Expand by the formula, $(x + 2)^{\frac{1}{2}}$.
13. Expand by the formula, $(ax + by)^5$.
14. Expand by the formula, $(ax + by)^{\frac{1}{2}}$.
15. Expand by the formula, $(ax + by)^{-\frac{1}{2}}$.
16. Expand by the formula, $(5a - 7x)^3$.
17. Expand by the formula, $(3a + 4x)^{\frac{1}{3}}$.
18. Find the n^{th} term of the development of $(ax + b)^{\frac{1}{2}}$.
19. Find the m^{th} term of the development of $(m - x)^m$.
20. Find the n^{th} term of the development of $(a - x)^{-\frac{1}{2}}$.

NOTE.—The student will observe that any function of x which can be developed into a series of ascending powers of x, may be developed by the same method. The general formula for this is called, from its originator, McLaurin's formula. (See p. 306, Note.)

CHAPTER XVIII.

LOGARITHMS.

434. The ***Logarithm*** of a number is the *measure of its factors.*

In other words, it is the *exponent* which shows how many times *the number* contains *a given number* as a *factor.*

435. Heretofore, when we have spoken of the *measure* of a quantity, the *measure* of its *terms* has been meant.

If we wish to find the measure of a line 27 feet long, with the yard as a unit, we may do it in either of three ways:

1st. Subtract 3 feet (the length of the unit) from 27, and again from the remainder, and continue so to do until *nothing is left.* The *number of subtractions* will be the *measure of the **terms*** of 27.

2d. Add 3+3+3, etc., until the *sum* is 27, and the *number of times* 3 is used will be the same measure.

3d. Divide 27 by 3, and the *quotient* will be the measure.

We may represent the three processes as follows:

1st. $27 - 3 - 3 - 3 - 3 - 3 - 3 - 3 - 3 - 3 = 0.$

2d. $3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 = 27.$

3d. $27 \div 3 = 9.$

Therefore 9 is the measure of 27 as a *term* when 3 is taken as the *unit term;* that is, 3 taken 9 times as a term equals 27.

We express this by using 9 as a coefficient, thus $9 \cdot 3$, or if we let y represent the yard,

Then, $$9y = 27 \text{ ft.}$$

436. In a similar manner we may ***measure the factors*** of a number. The measure of the *terms* of a number is the measure of its effect when added or subtracted, while

the measure of its *factors* measures its effect when used as a *multiplier* or *divisor*.

To find this measure we must assume a *unit factor*, as in measuring the terms we assumed a *unit term*.

Take the same number 27 to measure its factors, and assume 3 as the unit factor. As before, we may find the required measure in three ways:

1st. Take the factor 3 from 27, and again from the remaining factors, and so on until no factor remains. The number of times we can remove the factor 3 from 27 will be the measure required.

2d. Use 3 as a multiplier until the product equals 27; and the number of times 3 is used will be the measure.

3d. By a process to be explained hereafter, this measure can be found.

The first two processes we may represent thus:

$$\text{1st.} \quad [(27 \div 3) \div 3] \div 3 = 1.$$

$$\text{2d.} \quad 3 \times 3 \times 3 = 27.$$

In the *first* case we have taken the factor 3 from 27 three times and 1 only is left, which is a factor of no power, and occupies the same place with reference to factors that zero does to terms. (Art. 152.)

In the *second* case we find that 3 used three times as a factor equals 27.

Both give us 3 as the measure of the *factors* of 27 with 3 as the *unit factor*.

This is expressed by using an exponent thus, $3^3 = 27$.

437. The measure of the *factors* of a quantity is expressed by an ***Exponent;*** the measure of *terms* by a ***Coefficient.***

438. When in any mathematical computation only one *unit term* is employed, the *symbol* for that term is omitted, and the *coefficient* or *measure* only is written.

Thus, the surveyor who measures all his distances with the rod as a unit, does not write 10 *r*, 15 *r*, 20 *r*, etc., but simply 10, 15, 20, etc. Yet, in his computations he bears in mind the fact that the symbol for *rod* has been omitted, for when he multiplies two of these measures together, as 15×10, he calls it $150\ r^2$, or 150 square rods.

439. In a similar manner, computations in which numbers are used as *factors* may be abbreviated by measuring the factors of each with the same *unit factor*, and omitting that factor, writing only the *exponents* which express the measures of the factors. *Exponents* used in this way, as we have seen by the definition, are called ***Logarithms***. (Art. 434.)

Thus, if we adopt 2 as the unit factor, we have 2 as the measure of the factors of 4, 3 as the measure of the factors of 8, and 4 of 16, and so on. That is, we have:

$2^1 = 2$	$2^6 = 64$	$2^0 = 1$	$2^{-5} = \frac{1}{32}$
$2^2 = 4$	$2^7 = 128$	$2^{-1} = \frac{1}{2}$	$2^{-6} = \frac{1}{64}$
$2^3 = 8$	$2^8 = 256$	$2^{-2} = \frac{1}{4}$	$2^{-7} = \frac{1}{128}$
$2^4 = 16$	$2^9 = 512$	$2^{-3} = \frac{1}{8}$	$2^{-8} = \frac{1}{256}$
$2^5 = 32$	$2^{10} = 1024$	$2^{-4} = \frac{1}{16}$	$2^{-9} = \frac{1}{512}$

NOTE.—The abbreviation "log." is used to express logarithms. Thus, log. 16 indicates the logarithm of 16, and is read "logarithm of 16."

440. The factor adopted as the *unit factor* is called the ***Base***. This may be any number except 1.

That 1 cannot be used as the unit factor in measuring the factors of other quantities is evident, since 1 as a factor has no power, and it would require an infinite number of such factors to produce any number greater than 1.

For the same reason we cannot use zero for the unit of measure for terms.

441. In *measuring the factors* of numbers in this manner, the signs of the numbers are not and cannot be considered. We measure only the *factors of magnitude* or *value*, and not the *factors of direction*. Hence *the base* or *unit of measure* for these factors is a *number without a sign*.

442. But the *factors measured* may be used as *multipliers* or *divisors*, and the measure should indicate this. As the *measure of terms* is either + or —, according as the terms are used in *addition* or *subtraction*, so the *measure of factors* (logarithms) are either + or —, according as the *factors* are used as *multipliers* or *divisors*.

For example, if we measure the factors of 8 with 2 as a unit factor, the measure is $+3$; but if the 8 be used as a divisor, making it $\frac{1}{8}$, the measure is -3.

This will be further illustrated by referring to Art. 439. We have from that, when 2 is the base,

log. $2 = 1$	log. $1 = 0$
" $4 = 2$	" $\frac{1}{2} = -1$
" $8 = 3$	" $\frac{1}{4} = -2$
" $16 = 4$	" $\frac{1}{8} = -3$
	" $\frac{1}{16} = -4$

If now we make 2^{-1} $(= \frac{1}{2})$ the base, we have,

log. $2 = -1$	log. $1 = 0$
" $4 = -2$	" $\frac{1}{2} = 1$
" $8 = -3$	" $\frac{1}{4} = 2$
" $16 = -4$	" $\frac{1}{8} = 3$, etc., etc.

If we take the base 5, we have,

log. $1 = 0$	log. $1 = 0$
" $5 = 1$	" $\frac{1}{5} = -1$
" $25 = 2$	" $\frac{1}{25} = -2$
" $125 = 3$	" $\frac{1}{125} = -3$

If 5^{-1} $(= \frac{1}{5})$ be taken as the base, then

log. $1 = 0$	log. $1 = 0$
" $5 = -1$	" $\frac{1}{5} = 1$
" $25 = -2$	" $\frac{1}{25} = 2$
" $125 = -3$	" $\frac{1}{125} = 3$

443. We see that log. 1 will evidently be 0 for all bases, since 1 as a factor has no power, and its measure with any unit factor must be 0. So also the logarithm of the base will always be 1, since the measure of any quantity with itself as a unit must be 1.

444. The logarithms given above are all *integral*, because we have selected such numbers as contained the base as a factor an integral number of times.

Intermediate numbers have *intermediate* logarithms.

This may be illustrated by taking 16 as a base. We shall then have,

log. $1 = 0$	log. $1 = 0$
" $2 = \frac{1}{4} = .25$	" $\frac{1}{2} = -\frac{1}{4} = -.25$
" $4 = \frac{1}{2} = .5$	" $\frac{1}{4} = -\frac{1}{2} = -.5$
" $8 = \frac{3}{4} = .75$	" $\frac{1}{8} = -\frac{3}{4} = -.75$
" $16 = 1$	" $\frac{1}{16} = -1$
" $32 = \frac{5}{4} = 1.25$	" $\frac{1}{32} = -\frac{5}{4} = -1.25$
" $64 = \frac{3}{2} = 1.5$	" $\frac{1}{64} = -\frac{3}{2} = -1\ 5$

The logarithms of numbers between these are *incommensurable*, and can be expressed only by *approximation*.

445. The logarithms of numbers with a *given base* are called a *system of logarithms.**

The system in common use has 10 for a base, and from its originator is called ***Briggs' System.*** In this system,

log. $1 = 0$	log. $.1 = -1$
" $10 = 1$	" $.01 = -2$
" $100 = 2$	" $.001 = -3$
" $1000 = 3$	" $.0001 = -4$

The logarithms of numbers

between	1	and	10	are	0 +	a decimal.
"	10	"	100	"	1 +	"
"	100	"	1000	"	2 +	"
"	1	"	.1	"	−1 +	"
"	.1	"	.01	"	−2 +	"
"	.01	"	.001	"	−3 +	"

446. The *integral part* of a logarithm is called its ***Characteristic,*** and the *decimal part* its ***Mantissa.*** It is customary in expressing negative logarithms to make the mantissa positive, as indicated above.

Thus, if we have the logarithm -2.754526 (the whole being negative), it may be expressed thus,

$$-3 + .245474,$$

or, as it is commonly written,

$$\bar{3}.245474,$$

in which the 3 only is negative, the sign − being placed over it to indicate this.

* The method of computing by logarithms was invented by Lord Napier. (See p. 306, Notes 3 and 4.)

447. It will readily appear that the *characteristic* of the logarithm of a number in the common system may be known by the following

RULE.—*The number of places from the first significant figure of a number to units' place, counting the latter, is equal to the characteristic of the common logarithm of that number. When counted to the right, the characteristic is positive; when counted to the left, negative.*

NOTE.—The reason for this rule is found in the fact that multiplying or dividing a number by 10 moves the decimal point *one place* to the *right* or *left*, and at the same time increases or diminishes its logarithm one unit. Hence, moving the decimal point of a number to the right or left does not affect the mantissa of its logarithm.

448. What are the characteristics of the logarithms of the following numbers:

1.	2785.62.	*Ans.* 3.	
2.	54371.	*Ans.* 4.	
3.	.0075.	*Ans.* — 3.	
4.	1.075.		
5.	16.0005.		
6.	.0000589.	10.	5.89.
7.	0.00589.	11.	589.
8.	0.0589.	12.	58900.
9.	0.589.	13.	5890000.

449. When the number has integral figures,

The characteristic of its logarithm is one less than the number of integral places, and is positive.

450. When the number is entirely decimal,

The characteristic of its logarithm is one more than the number of ciphers between the decimal point and the first significant figure, and is negative.

TABLES OF LOGARITHMS.

451. A ***Table of Logarithms*** is one which contains the logarithms of all numbers between given limits.

452. The Table found on the following pages gives the mantissas of common logarithms to five decimal places for all numbers from 1 to 1000, inclusive.

The *characteristics* are omitted, and must be supplied by inspection. (Arts. 447–450.)

NOTES.—1. The first decimal figure in column 0 is often the same for several successive numbers, but is printed only once, and is understood to belong to each of the blank places below it.

2. The character (•) shows that the figure belonging to the place it occupies has changed from 9 to 0, and through the rest of this line the first figure of the mantissa stands in the next line below.

453. To Find the *Logarithm* of any Number from 1 to 10.

RULE.—*Look for the given number in the first line of the table; its logarithm will be found directly below it.*

1. Find the logarithm of 7. *Ans.* 0.84510.
2. Find the logarithm of 9. *Ans.* 0.95424.

454. To Find the *Logarithm* of any Number from 10 to 1000, inclusive.

RULE.—*Look in the column marked N for the first two figures of the given number, and for the third at the head of one of the other columns.*

Under this third figure, and opposite the first two, will be found the last four decimal figures of the logarithm. The first one is found in the column marked 0.

To this decimal prefix the proper characteristic. (Art. 447.)

NOTE.—If the number has 4 or more figures, find the logarithm of the first three figures and add to it the product of the remaining figures considered as a decimal, by the tabular difference (from column D) opposite the logarithm of the first three figures.

3. Find the logarithm of 108. *Ans.* 2.03342.

4. Find the logarithm of 176. *Ans.* 2.24551.

5. Find the logarithm of 1999. *Ans.* 3.30085.

455. To Find the Logarithm of a *Decimal Fraction.*

RULE.—*Take out the logarithm of a whole number consisting of the same figures, and prefix to it the proper negative characteristic.* (Art. 450.)

NOTE.—If the number consist of an *integer* and a *decimal*, find the logarithm in the same manner as if *all* the figures were integers, and prefix the characteristic which belongs to the *integral* part. (Art. 449.)

6. What is the log. of 0.95? *Ans.* $\bar{1}.97772$.

7. What is the log. of 0.0125? *Ans.* $\bar{2}.09691$.

8. What is the log. of 0.0075? *Ans.* $\bar{3}.87506$.

9. What is the log. of 16.45? *Ans.* 1.21616.

10. What is the log. of 185.3? *Ans.* 2.26787.

456. To Find the *Number* belonging to a given Logarithm.

RULE.—I. *Find in the table the mantissa next less than the mantissa of the given logarithm, and the corresponding number will be the first three figures of the required number.*

II. *Subtract the mantissa found in the table from the mantissa of the given logarithm, and divide the remainder by the corresponding tabular difference, for the remaining figures of the required number.*

III. *Place the decimal point of this number as required by the characteristic of the given logarithm.* (Art. 447.)

NOTE.—If the characteristic of a logarithm be *negative*, the number belonging to it is a *fraction*, and as many ciphers must be prefixed to the number found in the table, as there are *units* in the characteristic *less* 1. (Art. 450.)

11. What number belongs to 2.17231? *Ans.* 148.7.
12. What number belongs to 1.25261? *Ans.* 17.89.
13. What number belongs to 3.27715? *Ans.* 1893.
14. What number belongs to 2.30963? *Ans.* 204.
15. What number belongs to 4.29797? *Ans.* 19858.29.
16. What number belongs to $\bar{1}.14488$? *Ans.* 0.1396.
17. What number belongs to $\bar{2}.29136$? *Ans.* 0.01956.
18. What number belongs to $\bar{3}.30928$? *Ans.* 0.002038.

457. Computations in which numbers are used as factors may be made by logarithms upon the following principles:

1°. *The sum of the logarithms of two numbers is equal to the logarithm of their product.*

Let a and c denote any two numbers, m and n their logarithms, and b the base.

Then	$b^m = a$
And	$b^n = c$
Multiplying,	$b^{m+n} = ac.$

2°. *The logarithm of the dividend diminished by the logarithm of the divisor is equal to the logarithm of the quotient.*

Let a and c denote any two numbers, m and n their logarithms, and b the base.

Then	$b^m = a$
And	$b^n = c$
Dividing,	$b^{m-n} = a \div c.$

458. To *Multiply* Numbers by their Logarithms.

RULE.—*Add the logarithms of the factors; the sum will be the logarithm of the product.* (Art. 457, 1°.)

NOTES.—1. If one or more of the characteristics be *negative*, make them positive by adding 10 to each, and reject as many 10's from the sum.

2. The sign of the product or quotient is determined as when multiplication and division are performed in the usual manner. (Art. 129.)

1. Required the product of 35 by 23.

SOLUTION.—The log. of 35 =		1.54407
" " " 23 =		1.36173
	Adding,	2.90580.

The corresponding number is 805, *Ans.*

2. What is the product of 109.3 by 14.17?
3. What is the product of -1.465 by -1.347?
4. What is the product of .074 by -1500?

459. To *Divide* by Logarithms.

RULE.—*From the logarithm of the dividend subtract the logarithm of the divisor; the difference will be the logarithm of the quotient.* (Art. 457, 2°.)

NOTE.—If either or both characteristics are negative, add 10 to each and the result will not be affected.

5. Required the quotient of 120 by 15.

SOLUTION.—The log. of 120 =	2.07918
" " " 15 =	1.17609
" " " quotient =	0.90309. *Ans.* 8.

6. What is the quotient of 12.48 by 0.16?
7. What is the quotient of .045 by 1.20?
8. What is the quotient of 1.381 by .096?
9. Divide -128 by -47.
10. Divide -186 by -0.064.
11. Divide -0.156 by -0.86.
12. Divide -0.194 by 0.042.

460. To *Involve* a Number by Logarithms.

RULE.—*Multiply the logarithm of the number by the exponent of the required power.*

NOTE.—1. This rule depends upon the principle that logarithms are the *exponents* of *powers*, and a power is involved by multiplying its exponent into the exponent of the required power.

2. Let the student remember that *power* includes also *root*. (Art. 253.)

13. What is the cube of 1.246?

SOLUTION.—The log. of 1.246 is 0.09551

Index of the required power is 3

Logarithm of power is 0.28653. *Ans.* 1.93435.

14. What is the fourth power of .135?

15. What is the tenth power of 1.42?

16. What is the twenty-fifth power of 1.234?

17. What is the square root of 1.69?

NOTE.—Log. 1.69 must be multiplied by $\frac{1}{2}$, or what is the same thing, divided by 2.

18. What is the cube root of 143.2?

19. What is the sixth root of 1.62?

20. What is the eighth root of 1549?

21. What is the tenth root of 1876?

461. If the characteristic of the logarithm be *negative*, and cannot be *divided* by the *index* of the required root without a *remainder*, add to it such a *negative* number as will make it *exactly divisible* by the *divisor*, and *prefix* an equal *positive* number to the decimal part of the logarithm.

22. It is required to find the cube root of .0164.

SOLUTION.—The log of .0164 is $\bar{2}.21484$.

Preparing the log., $3\,)\,\underline{\bar{3}+1.21484}$

$\bar{1}.40494$. *Ans.* 0.25406+.

23. What is the sixth root of .001624?

24. What is the seventh root of .01449?

25. What is the eighth root of .0001236?

TABLE

OF

COMMON LOGARITHMS.

N.	0	1	2	3	4	5	6	7	8	9	D.
	.	.00000	.30103	.47712	.60206	.69897	.77815	.84510	.90309	.95424	
10	.00000	0432	0860	1284	1703	2119	2531	2938	3342	3743	416
11	4139	4532	4922	5308	5690	6070	6446	6819	7188	7555	379
12	7918	8279	8636	8991	9342	9691	•037	•380	•721	1059	349
13	.11394	1727	2057	2385	2710	3033	3354	3672	3988	4301	322
14	4613	4922	5229	5534	5836	6137	6435	6732	7026	7319	301
15	7609	7898	8184	8469	8752	9033	9312	9590	9866	•140	281
16	.20412	0683	0952	1219	1484	1748	2011	2272	2531	2789	264
17	3045	3300	3553	3805	4055	4304	4551	4797	5042	5285	249
18	5527	5768	6007	6245	6482	6717	6951	7184	7416	7646	235
19	7875	8103	8330	8556	8780	9003	9226	9447	9667	9885	223
20	.30103	0320	0535	0750	0963	1175	1387	1597	1806	2015	212
21	2222	2428	2634	2838	3041	3244	3445	3646	3846	4044	203
22	4242	4439	4635	4830	5025	5218	5411	5603	5793	5984	193
23	6173	6361	6549	6736	6922	7107	7291	7475	7658	7840	185
24	8021	8202	8382	8561	8739	8917	9094	9270	9445	9620	178
25	9794	9967	•140	•312	•483	•654	•824	•993	1162	1330	171
26	.41497	1664	1830	1996	2160	2325	2488	2651	2813	2975	165
27	3136	3297	3457	3616	3775	3933	4091	4248	4404	4560	158
28	4716	4871	5025	5179	5332	5485	5637	5788	5939	6090	153
29	6240	6389	6538	6687	6835	6982	7129	7276	7422	7567	147
30	.47712	7857	8001	8144	8287	8430	8572	8714	8855	8996	143
31	9136	9276	9415	9554	9693	9831	9969	•106	•243	•379	138
32	.50515	0651	0786	0920	1055	1188	1322	1455	1587	1720	133
33	1851	1983	2114	2244	2375	2504	2634	2763	2892	3020	130
34	3148	3275	3403	3529	3656	3782	3908	4033	4158	4283	126
35	4407	4531	4654	4777	4900	5023	5145	5267	5388	5509	123
36	5630	5751	5871	5991	6110	6229	6348	6467	6585	6703	119
37	6820	6937	7054	7171	7287	7403	7519	7634	7749	7864	116
38	7978	8093	8206	8320	8433	8546	8659	8771	8883	8995	113
39	9106	9218	9329	9439	9550	9660	9770	9879	9988	•097	110
40	.60206	0314	0423	0531	0638	0746	0853	0959	1066	1172	108
41	1278	1384	1490	1595	1700	1805	1909	2014	2118	2221	105
42	2325	2428	2531	2634	2737	2839	2941	3043	3144	3246	102
43	3347	3448	3548	3649	3749	3849	3949	4048	4147	4246	100
44	4345	4444	4542	4640	4738	4836	4933	5031	5128	5225	98
45	5321	5418	5514	5610	5706	5801	5896	5992	6087	6181	95
46	6276	6370	6464	6558	6652	6745	6839	6932	7025	7117	93
47	7210	7302	7394	7486	7578	7669	7761	7852	7943	8034	91
48	8124	8215	8305	8395	8485	8574	8664	8753	8842	8931	89
49	9020	9108	9197	9285	9373	9461	9548	9636	9723	9810	88
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
50	.69897	9984	•070	•157	•243	•329	•415	•501	•586	•672	86
51	.70757	0842	0927	1012	1096	1181	1265	1349	1433	1517	85
52	1600	1684	1767	1850	1933	2016	2099	2181	2263	2346	83
53	2428	2509	2591	2673	2754	2835	2916	2997	3078	3159	81
54	3239	3320	3400	3480	3560	3640	3719	3799	3878	3957	80
55	.74036	4115	4194	4273	4351	4429	4507	4586	4663	4741	78
56	4819	4896	4974	5051	5128	5205	5282	5358	5435	5511	77
57	5587	5664	5740	5815	5891	5967	6042	6118	6193	6268	76
58	6343	6418	6492	6567	6641	6716	6790	6864	6938	7012	75
59	7085	7159	7232	7305	7379	7452	7525	7597	7670	7743	73
60	.77815	7887	7960	8032	8104	8176	8247	8319	8390	8462	72
61	8533	8604	8675	8746	8817	8888	8958	9029	9099	9169	71
62	9239	9309	9379	9449	9519	9588	9657	9727	9796	9865	69
63	9934	•003	•072	•140	•209	•277	•346	•414	•482	•550	68
64	.80618	0686	0754	0821	0889	0956	1023	1090	1158	1224	67
65	1291	1358	1425	1491	1558	1624	1690	1757	1823	1889	66
66	1954	2020	2086	2151	2217	2282	2347	2413	2478	2543	65
67	2607	2672	2737	2802	2866	2930	2995	3059	3123	3187	64
68	3251	3315	3378	3442	3506	3569	3632	3696	3759	3822	63
69	3885	3948	4011	4073	4136	4198	4261	4323	4386	4448	63
70	.84510	4572	4634	4696	4757	4819	4880	4942	5003	5065	62
71	5126	5187	5248	5309	5370	5431	5491	5552	5612	5673	61
72	5733	5794	5854	5914	5974	6034	6094	6153	6213	6273	60
73	6332	6392	6451	6510	6570	6629	6688	6747	6806	6864	59
74	6923	6982	7040	7099	7157	7216	7274	7332	7390	7448	59
75	7506	7564	7622	7680	7737	7795	7852	7910	7967	8024	58
76	8081	8138	8196	8252	8309	8366	8423	8480	8536	8593	57
77	8649	8705	8762	8818	8874	8930	8986	9042	9098	9154	56
78	9209	9265	9321	9376	9432	9487	9542	9597	9653	9708	55
79	9763	9818	9873	9927	9982	•037	•091	•146	•200	•255	55
80	.90309	0363	0417	0472	0526	0580	0634	0687	0741	0795	54
81	0849	0902	0956	1009	1062	1116	1169	1222	1275	1328	54
82	1381	1434	1487	1540	1593	1645	1698	1751	1803	1855	52
83	1908	1960	2012	2065	2117	2169	2221	2273	2324	2376	52
84	2428	2480	2531	2583	2634	2686	2737	2788	2840	2891	52
85	2942	2993	3044	3095	3146	3197	3247	3298	3349	3399	51
86	3450	3500	3551	3601	3651	3702	3752	3802	3852	3902	51
87	3952	4002	4052	4101	4151	4201	4250	4300	4350	4399	50
88	4448	4498	4547	4596	4645	4694	4743	4792	4841	4890	49
89	4939	4988	5036	5085	5134	5182	5231	5279	5328	5376	48
90	.95424	5472	5521	5569	5617	5665	5713	5761	5809	5856	48
91	5904	5952	6000	6047	6095	6142	6190	6237	6284	6332	47
92	6379	6426	6473	6520	6567	6614	6661	6708	6755	6802	47
93	6848	6895	6942	6988	7035	7081	7128	7174	7220	7267	46
94	7313	7359	7405	7451	7497	7543	7589	7635	7681	7727	46
95	7772	7818	7864	7909	7955	8000	8046	8091	8137	8182	45
96	8227	8272	8318	8363	8408	8453	8498	8543	8588	8632	45
97	8677	8722	8767	8811	8856	8900	8945	8989	9034	9078	45
98	9123	9167	9211	9255	9300	9344	9388	9432	9476	9520	44
99	9564	9607	9651	9695	9739	9782	9826	9870	9913	9957	43
N.	0	1	2	3	4	5	6	7	8	9	D.

COMPUTATION OF LOGARITHMS.

Theorem I.

462. *In any two systems of logarithms, the logarithms of like numbers have a constant ratio.*

Demonstration.—Let the bases of the systems be a and a', and let x be any number whose logarithms in the two systems are z and z'. That is,

$$a^z = x \qquad \text{and} \qquad a'^{z'} = x.$$

$$\therefore \quad a^z = a'^{z'}. \qquad (1)$$

Let $\qquad a^m = a'.$

Substituting in (1), $\qquad a^z = a^{mz'}.$

$$\therefore \quad z = mz' \qquad \text{or} \qquad \frac{z}{z'} = m,$$

But since a and a' are constant, m is also constant.

Cor. 1.—*The logarithm of a number consists of two factors, one of which is a function of the base and the other a function of the number.*

This is evident, since changing the base introduces or removes a constant factor and makes no other change in the logarithm.

463. The ***Modulus of a System*** is this *constant factor*, depending on the base, and is usually represented by M.

The logarithm of x will therefore be $Mf(x)$, in which M is a function of the base of the system and is therefore constant.

Cor. 2.—*The base may be so chosen as to make the modulus* 1.

For in the equation $a^m = a'$, it is evident that a' may be so taken as to give m any value whatever; m may therefore be made such as to cancel the factor M, or modulus; so that, in $z = mz'$, the constant factor of z' shall be 1.

464. Baron Napier, who first suggested this use of exponents, took such a base for his system of logarithms, and they are called from the inventor the Napierian system.

The ***Napierian Logarithm*** of x will therefore be simply $f(x)$, the modulus being 1.

NOTE.—The Napierian system of logarithms is sometimes called the *natural system*, on account of its relation to other systems. Napierian logarithms are also called *hyperbolic logarithms*, by reason of their relations to certain areas connected with the hyperbola. The base of this system, commonly represented by e, is 2.718281 +. (Art. 468.)

465. Logarithms of different systems may be expressed by writing the *base* of the system *subscript* to the abbreviation log.

Thus, $\log_e$ indicates a Napierian, and $\log_{10}$ a common logarithm.

When no subscript figure or letter is used, the abbreviation log. must be understood to mean the common logarithm.

We may also write M_{10}, M_e, etc., for the moduli of the different systems.

COR. 3.—$\text{Log}_a x = M_a \log_e x$; *$M_a$ being the modulus of the system whose base is a.*

For $\log_a x = M_a f(x)$ **and** $\log_e x = f(x)$. **(Arts. 463–464.)**

It follows therefore that

466. *The modulus of any system is the ratio of any logarithm in that system to the Napierian logarithm of the same number.* Hence,

467. A table of logarithms with any base may be constructed by multiplying the logarithms of the Napierian system by the modulus of the required system.

COR. 4.—*The logarithms of the same number in different systems are to each other as their moduli.*

For $\log_a x = M_a f(x)$ **and** $\log_{a'} x = M_{a'} f(x)$.

$$\therefore \frac{\log_a x}{\log_{a'} x} = \frac{M_a}{M_{a'}}.$$

THEOREM II.

468. *The differential of the logarithm of a variable is equal to the modulus of the system multiplied by the differential of the variable divided by the variable.*

Let $$u = \log_a x. \qquad (1)$$

Adding dx to x, $u + du = \log_a (x + dx)$.

Subtracting (1), $$du = \log_a\left(1 + \frac{dx}{x}\right) = M_a f\left(1 + \frac{dx}{x}\right). \qquad (2)$$

(Arts. 457, 2°, and 463.)

To find $f\left(1 + \frac{dx}{x}\right)$, let x_1 and x_2 be any two values of x.

We may then write $$x_1 = x_2^n, \qquad (3)$$

And $$\log_a x_1 = n \log_a x_2. \qquad (4)$$

Differentiating (3), $$dx_1 = nx_2^{n-1}dx_2. \qquad (5)$$

Differentiating (4), $$d(\log_a x_1) = nd(\log_a x_2), \qquad (6)$$

(5) ÷ (3), $$\frac{dx_1}{x_1} = n\frac{dx_2}{x_2}. \qquad (7)$$

(6) ÷ (7), $$\frac{d(\log_a x_1)}{\frac{dx_1}{x_1}} = \frac{d(\log_a x_2)}{\frac{dx_2}{x_2}}.$$

That is, (Arts. 377–380), $$du \propto \frac{dx}{x}. \quad \therefore \quad du = m\frac{dx}{x}. \qquad (8)$$

From (2) and (8), $$m\frac{dx}{x} = M_a f\left(1 + \frac{dx}{x}\right).$$

Hence (Art. 419, Cor.), $$m = M_a,$$

And $$\frac{dx}{x} = f\left(1 + \frac{dx}{x}\right).$$

Hence $$du = M_a\frac{dx}{x}. \qquad (9)$$

If $a = e$, $$du = d(\log_e x) = \frac{dx}{x}. \qquad (10)$$

468*a*. To find the value of e, we have from (2), substituting e for a,

$e^{du} = 1 + \frac{dx}{x}$. For convenience put $du \left(= \frac{dx}{x}\right) = \frac{1}{n}$.

Then $$e^{\frac{1}{n}} = 1 + \frac{1}{n}. \qquad (11)$$

Raising (11) to the nth power,

$$e = 1 + n\frac{1}{n} + \frac{n(n-1)}{1 \cdot 2}\cdot\frac{1}{n^2} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}\cdot\frac{1}{n^3} + \text{etc.}$$

Since $\frac{1}{n} = \frac{dx}{x}$, an infinitesimal, $n = \infty$, and

$$e = 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \text{etc. (Art. 412.)} \qquad (12)$$

$$\therefore \quad e = 2.718281828459045+.$$

469. To find a *formula* for computing the Napierian logarithms of numbers, assume

$$\log_e(1+x) = A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc.} \qquad (1)$$

Differentiating successively, and removing factors common to both members.

$$\frac{1}{1+x} = B + 2Cx + 3Dx^2 + 4Ex^3 + \text{etc.} \qquad (2)$$

$$-\frac{1}{(1+x)^2} = 2C + 2\cdot 3Dx + 3\cdot 4Ex^2 + \text{etc.} \qquad (3)$$

$$+\frac{1}{(1+x)^3} = 3D + 3\cdot 4Ex + \text{etc.} \qquad (4)$$

$$-\frac{1}{(1+x)^4} = 4E + \text{etc.} \qquad (5)$$

Hence, making $x = 0$,

From (1),	$A = 0$,
" (2),	$B = 1$,
" (3),	$C = -\frac{1}{2}$,
" (4),	$D = \frac{1}{3}$,
" (5),	$E = -\frac{1}{4}$, etc.

Substituting in (1), and continuing the series by the law which becomes evident,

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \text{etc.} \qquad (6)$$

470. Formula (6) is called the ***Logarithmic Series***, but it is not in a form suited to the computation of logarithms, as may be seen by substituting 6 for x, which gives,

$$\log_e 7 = 6 - \frac{6^2}{2} + \frac{6^3}{3} - \frac{6^4}{4} + \text{etc.},$$

in which the terms are increasing, and it is impossible to take any number of terms which will give an approximate value of $\log_e 7$.

471. The following transformations will adapt this formula to the computation of the Napierian logarithms of numbers.

Putting $-x$ for x in (6),

$$\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \text{etc.} \qquad (7)$$

Subtracting (7) from (6),

$$\log_e(1+x) - \log_e(1-x) = \log_e\left(\frac{1+x}{1-x}\right)$$

$$= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \text{etc.}\right)$$

In this equation, making $x = \dfrac{1}{2z+1}$, we have, by writing $\log_e(1+z) - \log_e z$ for $\log_e\left(\dfrac{1+z}{z}\right)$, and transposing $\log_e z$,

$$\log_e(1+z) = \log_e z + 2\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \text{etc.}\right) \qquad \text{(A)}$$

472. To compute $\log_e$ for the numbers 1, 2, 3, 4, etc., we have, by Art. 443,

$$\log_e 1 = 0, \qquad (1)'$$

and by formula (A), making $z = 1$,

$$\log_e(1+1) = \log_e 2$$

$$= 0 + 2\left(\frac{1}{3} + \frac{1}{3\cdot 3^3} + \frac{1}{5\cdot 3^5} + \frac{1}{7\cdot 3^7} + \text{etc.}\right) \qquad (2)'$$

Making $z = 2$,

$$\log_e 3 = \log_e 2 + 2\left(\frac{1}{5} + \frac{1}{3\cdot 5^3} + \frac{1}{5\cdot 5^5} + \frac{1}{7\cdot 5^7} + \text{etc.}\right) \qquad (3)'$$

$$\log_e 4 = 2\log_e 2. \quad \text{(Art. 457, 1°.)} \qquad (4)'$$

Making $z = 4$,

$$\log_e 5 = \log_e 4 + 2\left(\frac{1}{9} + \frac{1}{3\cdot 9^3} + \frac{1}{5\cdot 9^5} + \frac{1}{7\cdot 9^7} + \text{etc.}\right) \qquad (5)'$$

473. To add a sufficient number of terms of series (2)′ to give $\log_e 2$ to six places of decimals, proceed as follows:

3)	2.00000000					
9)	.666666666	÷	1	=	.66666667	1st term.
9)	.07407407	÷	3	=	.02469136	2d "
9)	.00823045	÷	5	=	.00164609	3d "
9)	.00091449	÷	7	=	.00013064	4th "
9)	.00010161	÷	9	=	.00001129	5th "
9)	.00001129	÷	11	=	.00000103	6th "
9)	.00000125	÷	13	=	.00000009	7th "
	.00000014	÷	15	=	.00000001	8th "
			$\log_e 2$	=	.69314718	

NOTE.—In this computation the terms should be carried *two places* of decimals farther than the logarithm is to be used, to insure accuracy in the last figure.

From (3)′ in like manner we have,

5)	2.00000000					
25)	.40000000	÷	1	=	.40000000	1st term.
25)	.01600000	÷	3	=	.00533333	2d "
25)	.00064000	÷	5	=	.00012800	3d "
25)	.00002560	÷	7	=	.00000366	4th "
	.00000102	÷	9	=	.00000011	5th "
			$\log_e 2$	=	.69314718	
			$\log_e 3$	=	1.09861228	

From (4)′ we have,

$$\log_e 4 = 2 \log_e 2 = 1.38629436.$$

Since the sum of the logarithms of several factors equals the logarithm of their product, we need only compute by (A) the logarithms of the prime numbers.

Let the student find as above $\log_e 5$, $\log_e 6$, and $\log_e 7$.

474. We have $\log_a x = M_a \log_e x$. (Art. 465, Cor. 3.)

$$\therefore \quad M_a = \frac{\log_a x}{\log_e x},$$

in which, if x be made a,

$$M_a = \frac{\log_a a}{\log_e a} = \frac{1}{\log_e a}. \quad \text{Hence,}$$

The modulus of any system is the reciprocal of the Napierian logarithm of the base of the system.

474. *a.* Substituting $a-1$ for x in (6), Art. 469, we have,

$$\log_e a = a - 1 - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \tfrac{1}{4}(a-1)^4 + \text{etc.} \quad (8)$$

$$\therefore M_a = \frac{1}{\log_e a} = \frac{1}{a-1-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\frac{1}{4}(a-1)^4+\text{etc.}} \quad \ldots \quad (9)$$

and
$$M_{10} = \frac{1}{9 - \frac{9^2}{2} + \frac{9^3}{3} - \frac{9^4}{4} + \text{etc.}}$$

We also have, by substituting x for a in (8),

$$\log_e x = x - 1 - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \tfrac{1}{4}(x-1)^4 + \text{etc.} \quad (10)$$

Multiplying (10) by (9) gives,

$$\log_a x = \frac{x-1-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\frac{1}{4}(x-1)^4+\text{etc.}}{a-1-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\frac{1}{4}(a-1)^4+\text{etc.}}.$$

Representing this numerator by $f(x)$, the denominator will be $f(a)$, and

$$\log_a x = \frac{f(x)}{f(a)}.$$

Let the student show that

The modulus of the common system, M_{10}, is .43429448+.

475. To Compute a Table of Common Logarithms.

Multiply the Napierian logarithms by $M_{10} = .43429448+$. (Art. 467.)

CHAPTER XIX.

SERIES.

476. A ***Series*** is a *succession of quantities* which *increase* or *decrease* in accordance with some fixed law. These quantities are called the ***Terms*** of the series. For example,

Suppose a quantity x beginning with the value 1 to increase in such manner as to double each second. If we write the values of x taken at equal intervals of time, they will form a series. If taken at intervals of one second, the series will be

$$1, \quad 2, \quad 4, \quad 8, \quad 16, \quad 32, \quad \text{etc.},$$

or if taken at intervals of two seconds,

$$1, \quad 4, \quad 16, \quad 64, \quad \text{etc.}$$

Again, suppose the side of a square, beginning at 0, to increase uniformly at the rate of one inch per second. If the area of the square be taken each second, we have the series,

$$1^2, \quad 2^2, \quad 3^2, \quad 4^2, \quad 5^2, \quad \text{etc.}$$

If once in two seconds, we have,

$$1^2, \quad 3^2, \quad 5^2, \quad 7^2, \quad \text{etc.}$$

477. The ***Law of a Series*** must therefore express two things.

1st. *The rate of increase of the quantity.*

2d. *The intervals of time at which its values are taken for the terms of the series.*

The law of a series is usually expressed in the form of a general rule for the formation of any term from the preceding term or terms; or we may have several terms of a series given from which to determine the law,

478. Since there is *no limit* to the number of different laws which may govern the formation of series, there will be an *unlimited variety* of series. Our space will only allow the discussion of a few of the most important.

479. For convenience we number the terms of a series from left to right, beginning with some term which we call the first term; but as it is evident that any series may be extended both ways, we shall not only have terms numbered 1, 2, 3, 4, 5, etc., to the right, but also those numbered 0, —1, —2, —3, —4, etc., to the left.

480. The problems to be discussed relating to series are,

1st. *Finding any required term of a series.*

2d. *Interpolation of terms.*

3d. *Summation of series.*

4th. *Reversion of series.*

481. ***To find any term of a series*** requires a formula which will give the value of the variable quantity at any given time; as in the series of the squares, to find the 10th term is the same thing as finding the area of the square at the end of 10 seconds.

482. ***Interpolation of terms*** is the process of finding one or more terms intermediate between any two terms of a series.

Thus, if we find the area of the square (Art. 476) at the end of $5\frac{1}{2}$ seconds, we shall have a term of the series between the 5th and 6th, and equidistant from each; that is, equidistant *in time*, but *not in value*.

483. The practical value of this problem may be illustrated by supposing the altitude of the sun to be known for noon, and for each hour after noon till sunset. These altitudes will form a series, and if the law can be found we can find the sun's altitude for any intermediate time, say for $2\frac{1}{4}$, $2\frac{1}{2}$, and $2\frac{3}{4}$ o'clock. These will form three terms between the *third* and *fourth* terms of the series.

484. It is evident that the formula for finding any term of a series will apply to interpolation, for if we can find the value of a variable at the end of 10 seconds, we can by the same process find it for $9\frac{1}{2}$ or $9\frac{3}{4}$ seconds.

485. The ***Summation of series*** is the process of finding the sum of any number of terms of a series. The method will of course depend on the *law* of the series.

486. A ***Converging Series*** is one in which the sum of an infinite number of terms is *finite.*

487. A ***Diverging Series*** is one in which the sum of an infinite number of terms is *infinite.*

488. A series is *increasing* or *decreasing* according as its successive terms *increase* or *decrease.*

DIFFERENCE SERIES.

489. Take the series

$$1 \;.\; 5 \;.\; 15 \;.\; 35 \;.\; 70 \;.\; 126 \;.\; 210. \qquad (1)$$

By subtracting the *first term* from the *second*, the *second* from the *third*, and so on, we have a series of differences called the *first order of differences.*

From these differences another set of differences may be formed in the same way, called the *second order of differences*, and from these the *third order*, and so on till the differences become zero.

The series (1) and its several orders of differences will be as follows:

Series,	1		5		15		35		70		126		210
1st order of diff.		4		10		20		35		56		84	
2d order of diff.			6		10		15		21		28		
3d order of diff.				4		5		6		7			
4th order of diff.					1		1		1				
5th order of diff.						0		0					

490. A series which, like the above, gives an *order* of *differences* equal to *zero*, is called a ***Difference Series.*** Not only is series (1) a difference series, but each set of differences is also a difference series.

The series 4, 5, 6, 7, etc., having its first differences constant (1 . 1 . 1, etc.), is called a *difference series of the first order;* 6 . 10 . 15, etc., having the second differences constant, is of the *second order.* So also 4 . 10 . 20, etc., is of the *third order*, and 1 . 5 . 15 . 35, etc., of the *fourth order.*

It is also evident that this series may be made the differences of a series of the fifth order, and so on indefinitely.

491. A difference series of the *first order,* is called an ***Equidifferent Series,*** because each term is formed by adding a constant difference to the term preceding.

NOTE.—This series is commonly called an *Arithmetical Series* or *Progression.*

492. To find formulas for the n^{th} term and the sum of n terms of a difference series of any order, take the series,

$$a_1 \qquad a_2 \qquad a_3 \qquad a_4 \qquad a_5 \qquad a_6$$

1st order of differences,

$$a_2-a_1 \qquad a_3-a_2 \qquad a_4-a_3 \qquad a_5-a_4 \qquad a_6-a_5$$

2d order of differences,

$$a_3-2a_2+a_1 \qquad a_4-2a_3+a_2 \qquad a_5-2a_4+a_3 \qquad a_6-2a_5+a_4$$

3d order of differences,

$$a_4-3a_3+3a_2-a_1 \qquad a_5-3a_4+3a_3-a_2 \qquad a_6-3a_5+3a_4-a_3$$

4th order of differences,

$$a_5-4a_4+6a_3-4a_2+a_1 \qquad a_6-4a_5+6a_4-4a_3+a_2$$

5th order of differences,

$$a_6-5a_5+10a_4-10a_3+5a_2-a_1.$$

If we put the first terms of these successive orders of differences $=$ d_1, d_2, d_3, etc., we shall have,

$$d_1 = a_2-a_1.$$
$$d_2 = a_3-2a_2+a_1.$$
$$d_3 = a_4-3a_3+3a_2-a_1.$$
$$d_4 = a_5-4a_4+6a_3-4a_2+a_1.$$
$$d_5 = a_6-5a_5+10a_4-10a_3+5a_2-a_1.$$

In these equations we find the coefficients are the same as for the n^{th} power of a binomial.

From this we can write

$$d_n = a_{n+1} - na_n + \frac{n(n-1)}{1 \cdot 2} a_{n-1} - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a_{n-2} + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} a_{n-3} - \text{etc.}$$

Reversing the order of terms,

$$d_n = \pm a_1 \mp na_2 \pm \frac{n(n-1)}{1 \cdot 2} a_3 \mp \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a_4 \pm \text{etc.},$$

in which the *upper* signs will be used when n is *even*, and the *lower* when n is *odd*.

From the values of d_1, d_2, etc., we get,

$$a_2 = a_1 + d_1.$$
$$a_3 = a_1 + 2d_1 + d_2.$$
$$a_4 = a_1 + 3d_1 + 3d_2 + d_3.$$
$$a_5 = a_1 + 4d_1 + 6d_2 + 4d_3 + d_4.$$

And from the law of the coefficients, which is evident,

$$a_n = a_1 + (n-1) d_1 + \frac{(n-1)(n-2)}{1 \cdot 2} d_2 + \frac{(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3} d_3 + \text{etc.} \quad (A)$$

493. From this formula any term of a difference series of any order may be found, when enough of its terms are known to give the first terms of the several orders of differences. The number of terms of the formula used in any case will depend on the order of the series. Thus for series of the first order (Equidifferent series), all the differences after d_1 will be *zero*. Hence the formula will become

$$a_n = a_1 + (n - 1) d \quad (A)'$$

corresponding to the common formula for the n^{th} term of an ***Arithmetical Series.***

NOTE.—The subscript 1 is omitted from d as unnecessary.

494. To find the ***Sum*** of n terms of the ***Difference Series,***

$$a_1, \quad a_2, \quad a_3, \quad a_4, \quad \text{etc.},$$

form another series of which the given series shall be the first order of differences; thus,

$$0, \quad a_1, \quad a_1 + a_2, \quad a_1 + a_2 + a_3, \quad \text{etc.}$$

It is evident that the $(n+1)^{th}$ *term* of this series is the sum of n *terms* of the given series; hence, if we apply formula (A) and find the $(n+1)^{th}$ term of this last series, we shall have the *sum of n terms* of the given series, as required. To make this application we must make in (A),

$$n = n+1, \quad a_1 = 0, \quad d_1 = a_1, \quad d_2 = d_1, \quad \text{etc.}$$

Making these substitutions, we have, putting $a_{n+1} = S_n$,

$$S_n = na_1 + \frac{n(n-1)}{1 \cdot 2} d_1 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} d_2 + \text{etc.} \qquad \text{(B)}$$

NOTE.—S is used for the sum of a series, with a subscript letter or figure to indicate the number of terms included.

495. If the series be ***Equidifferent,*** this becomes

$$S_n = na_1 + \frac{n(n-1)}{1 \cdot 2} d, \qquad \text{(B)}'$$

which, by substituting the value of d from (A)′, becomes

$$S_n = \frac{a_1 + a_n}{2} n. \qquad \text{(B)}''$$

This is the common formula for the sum of an ***Arithmetical Series.***

496. The formula for the n^{th} term of a series is also used for *interpolation.* (Art. 484.) In that formula n, which represents the *number* of any term, may be more properly regarded as representing the *time* at which the value of the variable is taken for any term. Hence the formula for the n^{th} term applies equally well when n is *fractional* as when it is *integral.*

If therefore it be required to interpolate 3 terms between the 9th and 10th terms of a series, we have only to make $n = 9\frac{1}{4}$, $9\frac{1}{2}$, and $9\frac{3}{4}$, successively in the formula for the n^{th} term of the series.

When the terms are known between which other terms are to be interpolated, the preceding terms of the series may be disregarded, and these two terms may be called the *first* and *second* terms of the series. One term to be interpolated will be the $1\frac{1}{2}$ term, two will be $1\frac{1}{3}$ and $1\frac{2}{3}$, three will be $1\frac{1}{4}$, $1\frac{1}{2}$, $1\frac{3}{4}$, and so on.

NOTE.—Terms thus interpolated are called *Means*, and the terms between which they are inserted are called *Extremes*.

497. When several *equidifferent means* are to be inserted between two extremes, it may be convenient to use a formula for d, obtained from Eq. (A)′; thus,

$$a_n = a_1 + (n - 1)\,d,$$

gives
$$d = \frac{a_n - a_1}{n - 1}; \qquad \text{(C)}$$

in which a_n and a_1 represent the two extremes, and n the number of terms in the *completed series*, or *two more* than the number to be interpolated.

Thus to insert 4 means between 5 and 20, we have

$$d = \frac{20 - 5}{6 - 1} = 3.$$

Hence the series complete is

$$5 \cdot 8 \cdot 11 \cdot 14 \cdot 17 \cdot 20.$$

By formula (A), a series may also be carried backward, calling the numbers of the terms from the first term,

$$0, \quad -1, \quad -2, \quad -3, \quad -4, \quad \text{etc.}$$

This may be illustrated by the series,

$$1 \cdot 4 \cdot 10 \cdot 20 \cdot 35. \quad \text{Thus,}$$

−5	−4	−3	−2	−1	0	1	2	3	4	5	6
−10	−4	−1	0	0	0	(1	4	10	20	35)	56
	6	3	1	0	0	1	(3	6	10	15)	21
	−3	−2	−1	0	1	2	(3	4	5)	6	
		1	1	1	1	1	1	(1	1)	1	

The first line shows the number of each term of the series. These terms are formed by extending the equidifferent series, and, from that, forming terms of the series of the second order, and then those of the third. We may also find any one of these terms by formula (A). (Art. 492.)

1. Given the series $1 \cdot 8 \cdot 27 \cdot 64 \cdot 125$. etc., to find: 1st, the 15th term; 2d, the sum of 15 terms; 3d, the first of three terms interpolated between the 5th and 6th; 4th, the -4th term; 5th, the n^{th} term; 6th, the sum of n terms.

SOLUTIONS.

$$\begin{array}{ccccccccccc}
1 & & 8 & & 27 & & 64 & & 125 & & 216 \\
 & 7 & & 19 & & 37 & & 61 & & 91 & \\
 & & 12 & & 18 & & 24 & & 30 & & \\
 & & & 6 & & 6 & & 6 & & & \\
 & & & & 0 & & 0 & & & &
\end{array}$$

1st. From formula (A) we have,

$$a_{15} = 1 + 14 \cdot 7 + \frac{14 \cdot 13}{2} 12 + \frac{14 \cdot 13 \cdot 12}{2 \cdot 3} 6;$$

$$\therefore \quad a_{15} = 1 + 98 + 1092 + 2184 = 3375.$$

2d. By formula (B),

$$S_{15} = 15 + \frac{15 \cdot 14}{2} 7 + \frac{15 \cdot 14 \cdot 13}{2 \cdot 3} 12 + \frac{15 \cdot 14 \cdot 13 \cdot 12}{2 \cdot 3 \cdot 4} 6;$$

$$\therefore \quad S_{15} = 15 + 735 + 5460 + 8190 = 14400.$$

3d. The first of three terms inserted between the fifth and sixth will be the $5\frac{1}{4}$ term.

Using formula (A), we have,

$$a_{5\frac{1}{4}} = 1 + 4\tfrac{1}{4} \cdot 7 + \frac{4\frac{1}{4} \cdot 3\frac{1}{4}}{2} 12 + \frac{4\frac{1}{4} \cdot 3\frac{1}{4} \cdot 2\frac{1}{4}}{2 \cdot 3} 6;$$

$$\therefore \quad a_{5\frac{1}{4}} = 1 + 29\tfrac{3}{4} + 82\tfrac{7}{8} + 31\tfrac{5}{64} = 144\tfrac{45}{64}.$$

4th. By the same formula the -4^{th} term is

$$a_{-4} = 1 + (-5)\,7 + \frac{(-5)(-6)}{2} 12 + \frac{(-5)(-6)(-7)}{2 \cdot 3} 6;$$

$$\therefore \quad a_{-4} = 1 - 35 + 180 - 210 = -64.$$

5th. The same formula gives the n^{th} term,

$$a_n = 1 + (n-1)\,7 + \frac{(n-1)(n-2)}{1 \cdot 2} 12 + \frac{(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3} 6;$$

$$\therefore \quad a_n = 1 + 7(n-1) + 6(n-1)(n-2) + (n-1)(n-2)(n-3) = n^3.$$

6th. The sum of n terms from formula (B) is

$$S_n = n + \frac{n(n-1)}{2}7 + \frac{n(n-1)(n-2)}{2 \cdot 3}12 + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 3 \cdot 4}6;$$

$$\therefore \quad S_n = n + \tfrac{7}{2}n(n-1) + 2n(n-1)(n-2) + \tfrac{1}{4}n(n-1)(n-2)(n-3)$$

$$= \left[\frac{n(n+1)}{2}\right]^2.$$

EXAMPLES.

Find the n^{th} *term* and the *sum* of n terms of the following series, and apply the formulas thus obtained by making $n =$ different numbers. Also interpolate terms until the formulas are familiar.

1. 1, 4, 7, 10, 13, etc.
2. 3, $6\frac{1}{2}$, 10, $13\frac{1}{2}$, 17, etc.
3. 2, 7, 12, 17, etc.
4. 2, 6, 13, 23, etc.
5. 1, 2, 3, 4, 5, etc.
6. 1, 3, 5, 7, 9, etc.
7. 2, 4, 6, 8, etc.
8. 1, 3, 6, 10, etc.
9. 1^2, 2^2, 3^2, 4^2, etc.
10. 1^3, 2^3, 3^3, 4^3, etc.
11. 1, 4, 10, 20, 35, etc.
12. 1, 5, 15, 35, 70, 126, etc.
13. 1, 6, 21, 56, 126, 252, 462, etc.

14. What is the 0 term, the -1st term, and the -2d term of Ex. 11 above?

15. Which of the above series gives the number of balls that can be piled in a pyramid whose base is an equilateral triangle? Which the number that can be piled in a pyramid whose base is a square?

16. How many balls can be piled in a triangular pyramid having 10 balls on each side of the lowest tier?

17. How many in a quadrangular pyramid, having the same number on each side in the lowest tier?

18. How many in an oblong rectangular pile 20 balls long and 5 balls wide?

19. Find the — 10th term of Ex. 12 above.

20. If a body fall 16 feet in one second, 3 times as far the next second, 5 times as far the third, and so on, how far will it fall the tenth second? How far in 10 seconds? How far in $7\frac{1}{2}$ seconds? How far in $5\frac{1}{3}$ seconds? How far in n seconds?

21. Find from

$$a_n = a_1 + (n - 1)d,$$

$$S_n = \frac{n(a_1 + a_n)}{2},$$

the following ***Formulas for Equidifferent Series*:**

$$a_1 = a_n - (n - 1)d. \qquad (1)$$

$$a_1 = \frac{2S_n}{n} - a_n. \qquad (2)$$

$$a_1 = \frac{S_n}{n} - \frac{(n - 1)d}{2}. \qquad (3)$$

$$a_1 = \frac{d}{2} \pm \sqrt{(a_n + \frac{1}{2}d)^2 - 2dS_n}. \qquad (4)$$

$$a_n = a_1 + (n - 1)d. \qquad (5)$$

$$a_n = \frac{2S_n}{n} - a_1. \qquad (6)$$

$$a_n = \frac{S_n}{n} + \frac{(n - 1)d}{2}. \qquad (7)$$

$$a_n = -\frac{d}{2} \pm \sqrt{2dS_n + (a_1 - \frac{1}{2}d)^2}. \qquad (8)$$

$$S_n = \frac{n(a_1 + a_n)}{2}. \qquad (9)$$

$$S_n = \frac{n}{2}[2a_1 + (n-1)d]. \qquad (10)$$

$$S_n = \frac{a_n + a_1}{2} + \frac{a_n^2 - a_1^2}{2d}. \qquad (11)$$

$$S_n = \frac{n}{2}[2a_n - (n-1)d]. \qquad (12)$$

$$d = \frac{a_n - a_1}{n - 1}. \qquad (13)$$

$$d = \frac{a_n^2 - a_1^2}{2S_n - a_n - a_1}. \qquad (14)$$

$$d = \frac{2(na_n - S_n)}{n(n-1)}. \qquad (15)$$

$$d = \frac{2(S_n - na_1)}{n(n-1)} \qquad (16)$$

$$n = \frac{a_n - a_1}{d} + 1. \qquad (17)$$

$$n = \frac{2S_n}{a_1 + a_n}. \qquad (18)$$

$$n = \frac{d - 2a_1 \pm \sqrt{(2a_1 - d)^2 + 8dS_n}}{2d}. \qquad (19)$$

$$n = \frac{2a_n + d \pm \sqrt{(2a_n + d)^2 - 8dS_n}}{2d}. \qquad (20)$$

RECURRING SERIES.

498. A ***Recurring Series*** is one in which each term is formed by multiplying the n preceding terms each by a *constant multiplier*, and adding the products. These multipliers are called the ***Scale of the Series.***

Thus, in the series

$$1, \quad 4, \quad 9, \quad 16, \quad 25,$$

each term may be formed by multiplying the three preceding terms by 1, -3, and $+3$, respectively, and adding the products; as,

$$1 \times 1 + 4(-3) + 9 \times 3 = 16,$$
$$4 \times 1 + 9(-3) + 16 \times 3 = 25,$$
$$9 \times 1 + 16(-3) + 25 \times 3 = 36.$$

499. This series is also a *difference series* of the *second order*, and any term or the sum of any number of terms may be found by the formulas already given.

500. There are, however, cases of recurring series to which the method of differences cannot be readily applied. These may be treated by finding the *scale of the series*, and using it to find the term or sum required.

501. To find the *scale of a recurring series*, assume the series

$$a_1, \quad a_2, \quad a_3, \quad a_4, \quad a_5, \quad a_6, \text{ etc.}$$

If each term depends on one preceding term only, we have

$$a_2 = ma_1,$$

in which m is the constant multiplier. This gives

$$m = \frac{a_2}{a_1}.$$

502. This is a *recurring series* of the *first order*, and is called an ***Equimultiple Series.***

It is also called a ***Geometrical Series*** or ***Progression,*** and is the most important of recurring series.

503. If each term of the series depends on two preceding terms, the series is of the second order, and we shall have

$$a_3 = m_1 a_1 + m_2 a_2,$$

and $$a_4 = m_1 a_2 + m_2 a_3,$$

from which m_1 and m_2, the scale of the series, are found to be

$$m_1 = \frac{a_2 a_4 - a_3^2}{a_2^2 - a_1 a_3}, \qquad m_2 = \frac{a_2 a_3 - a_1 a_4}{a_2^2 - a_1 a_3}.$$

504. In the same way the scale may be found when the series is of a higher order. If the order of the series be not known, we may assume it to be of a certain order, and find the scale.

If the order be taken too high, one or more of the multipliers will be zero, and the order and scale will be determined. If taken too low, the scale found will be incorrect, but the error will be discovered in applying the multipliers.

For example, to find the scale of

$$1, \quad 3, \quad 6, \quad 10, \quad 15, \quad 21, \quad \text{etc.},$$

assume the series to be of the second order.

We shall then have, $\quad m_1 + 3m_2 = 6,$

$$3m_1 + 6m_2 = 10.$$

From these equations we find $\quad m_1 = -2,$

$$m_2 = \tfrac{8}{3}.$$

In attempting to extend the series by means of this scale, we find it fails; but assuming the series to be of the third order,

$$m_1 + 3m_2 + 6m_3 = 10,$$
$$3m_1 + 6m_2 + 10m_3 = 15,$$
$$6m_1 + 10m_2 + 15m_3 = 21.$$

From which, $m_1 = 1$, $m_2 = -3$, $m_3 = 3$, which is the *true scale*.

The same would have been found if we had assumed the series to be of the fourth or any higher order.

EXAMPLES.

Find the scale of the following series:

1. $a_1, \quad a_1 + d, \quad a_1 + 2d, \quad a_1 + 3d,$ etc.

Ans. $-1, +2$.

2. $a_2, \quad a_2+a_1, \quad a_2+2a_1+d, \quad a_2+3a_1+3d, \quad a_2+4a_1+6d,$ $a_2 + 5a_1 + 10d,$ etc. *Ans.* $1, -3, +3$.

3. $a_3, \quad a_3 + a_2, \quad a_3 + 2a_2 + a_1, \quad a_3 + 3a_2 + 3a_1 + d,$ $a_3+4a_2+6a_1+4d, \quad a_3+5a_2+10a_1,+10d, \quad a_3+6a_2+15a_1+20d,$ $a_3 + 7a_2 + 21a_1 + 35d,$ etc. *Ans.* $-1, +4, -6, +4$.

4. $a_4, \quad a_4 + a_3, \quad a_4 + 2a_3 + a_2, \quad a_4 + 3a_3 + 3a_2 + a_1,$ $a_4 + 4a_3 + 6a_2 + 4a_1 + d, \quad a_4 + 5a_3 + 10a_2 + 10a_1 + 5d,$ $a_4 + 6a_3 + 15a_2 + 20a_1 + 15d, \quad a_4 + 7a_3 + 21a_2+35a_1+35d,$ $a_4+8a_3+28a_2+56a_1+70d, \quad a_4+9a_3+36a_2+84a_1+126d,$ etc.

Ans. $1, -5, +10, -10, +5$.

505. The student will observe that Examples 1 to 4 are general expressions for *difference series* of the 1st, 2d, 3d, and 4th orders respectively, of which

$$d, \quad d, \quad d, \quad d, \quad \text{etc.},$$

are the constant differences. These constant differences form a series whose scale is 1, and we may call it a *difference series of the zero order.*

506. From these solutions we infer the following principles:

1°. *Every difference series is also a recurring series.*

2°. *The order of a series as a difference series is one less than the order of the same series as a recurring series.*

3°. *All difference series of the same order have the same scale when regarded as recurring series.*

4°. *These scales are the same as the coefficients of the n^{th} power of $a - x$, omitting the first, changing their signs and reversing the terms; n being the order of the series considered as a recurring series.*

507. Any recurring series not having such a scale cannot be a *true difference* series; but when a series has an *order of differences* very nearly constant, the application of the formulas for difference series will give approximate results.

For illustration take the following example:

1. Given log. 200 = 2.30103,
   " " 210 = 2.32222,
   " " 220 = 2.34242,
   " " 230 = 2.36173,
   " " 240 = 2.38021,
   " " 250 = 2.39794,

to find log. 205.

Solution.—The given logarithms form a series, of which log. 205 is the $1\frac{1}{2}$ term. Finding the differences, we have,

Series,	2.30103	2.32222	2.34242	2.36173	2.38021	2.39794
1st diff.,	.02119	.02020	.01931	.01848	.01773	
2d "		−.00099	−.00089	−.00083	−.00075	
3d "			.00010	.00006	.00008	
4th "				−.00004	.00002	

By Formula (A),

$$a_{1\frac{1}{2}} = 2.30103 + \tfrac{1}{2}(.02119) + \frac{\frac{1}{2}\cdot\frac{1}{2}}{1\cdot 2}(.00099) + \frac{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}}{1\cdot 2\cdot 3}(.00010)$$
$$+ \frac{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{1\cdot 2\cdot 3\cdot 4}(.00004).$$

$$\therefore a_{1\frac{1}{2}} = \text{log. } 205 = 2.30103 + .010595 + .000124 + .000006 + .000002$$
$$= 2.31175+,$$

which agrees with the log. 205 from the table, although the series is not a perfect difference series.

2. Find in like manner log. 215.
3. Find in like manner log. 225.
4. Find in like manner log. 232.

508. The ***Equimultiple Series,*** which is a recurring series of the first order, and whose terms are each formed by multiplying the preceding term by a constant multiplier, may be written,

$$a_1, \quad a_1m, \quad a_1m^2, \quad a_1m^3, \quad \text{etc.},$$

in which a_1 is the first term and m the constant multiplier. This obviously gives,

$$a_n = a_1m^{n-1}. \qquad \text{(D)}$$

Also, $S_n = a_1 + a_1m + a_1m^2 \ldots\ldots + a_1m^{n-1}.$

Multiplying by m,

$$mS_n = a_1m + a_1m^2 + a_1m^3 \ldots\ldots + a_1m^n.$$

Subtracting the first from the second,

$$mS_n - S_n = a_1m^n - a_1,$$

and $$S_n = \frac{a_1m^n - a_1}{m - 1} = \frac{a_1(m^n - 1)}{m - 1}. \qquad \text{(E)}$$

509. If m be a positive integer, greater than unity, the series will be increasing; and if negative, the terms will increase numerically, but will be alternately + and —.

If m be a *proper* fraction, the series will decrease; and if the number of terms be infinite, a_n will be by Formula (D),

$$a_\infty = a_1m^\infty = 0,$$

and $$S_\infty = \frac{a_1(m^\infty - 1)}{m - 1}.$$

or $$S_\infty = \frac{a_1}{1 - m}. \qquad \text{(E)}'$$

510. The method of interpolation already given applies to all series when a formula for a_n can be found. (Art. 496.) It is therefore applicable to *equimultiple* series, and we may use for this purpose Formula (D).

511. To find a term between the 4th and 5th, make

$$n = 4\tfrac{1}{2}.$$

Then $$a_{4\frac{1}{2}} = a_1 m^{3\frac{1}{2}} \quad \text{or} \quad a_1 m^{\frac{7}{2}}.$$

To interpolate two terms between the 4th and 5th, n must be made $4\frac{1}{3}$ and $4\frac{2}{3}$ successively.

512. Interpolation may also be performed by finding the multiplier of the series formed by the interpolated and adjacent terms. This will abbreviate the work when many terms are to be interpolated.

Putting m' for this multiplier, and n' for the number of terms to be interpolated between any two terms, we have

$$m' = m^{\frac{1}{n'+1}}. \qquad \text{(F)}$$

Thus, to interpolate 3 terms between any two terms of the series,

$$1, \quad 16, \quad 256, \quad \text{etc.},$$

in which $$m = 16, \qquad m' = 16^{\frac{1}{3+1}} = 16^{\frac{1}{4}} = 2.$$

If these terms be inserted between 16 and 256, the series will be,

$$16, \quad 32, \quad 64, \quad 128, \quad 256.$$

EXAMPLES.

513. Perform the examples, p. 215, by the principles of recurring series, so far as they are applicable; also the following:

1. Find a_{20} and S_{20}, also a_n and S_n in

$$1, \quad 2, \quad 4, \quad 8, \quad 16, \quad \text{etc.}$$

2. Find a_{10} and S_{10}, also a_n and S_n in

$$3, \quad 9, \quad 27, \quad 81, \quad \text{etc.}$$

3. Find a_{10} and S_{10}, also a_n and S_n in

$$5, \quad 10, \quad 20, \quad 40, \quad \text{etc.}$$

4. Find a_∞ and S_∞ in

$$8, \quad 4, \quad 2, \quad 1, \quad \text{etc.}$$

5. Find a_∞ and S_∞ in

$$4, \quad 1, \quad \tfrac{1}{4}, \quad \text{etc.}$$

6. Find m' for interpolating a single term between any two of the last series, and find what the term between the 5th and 6th will be.

7. Interpolate two terms between each two of the series 1, 8, 64, etc., by finding m'.

8. Find a_{10} and S_{10}, also a_8 and S_8 of the series

$$1, \quad -3, \quad +9, \quad -27, \quad +\text{etc.}$$

9. Find the scale of

$$1, \quad 2x, \quad 3x^2, \quad 5x^3, \quad 10x^4, \quad 21x^5, \quad 43x^6,$$

and carry the series to the 10th term.

10. Find the scale and a_{12} of the series,

$$2, \quad 5x, \quad 8x^2, \quad 11x^3, \quad 14x^4, \quad 17x^5.$$

11. Find from

$$a_n = a_1 m^{n-1},$$

$$S_n = \frac{a_1(m^n - 1)}{m - 1},$$

the following ***Formulas for Equimultiple Series:***

$$a_1 = \frac{a_n}{m^{n-1}}. \qquad (1)$$

$$a_1 = a_n m - S_n(m-1). \qquad (2)$$

$$a_1 = \frac{(m-1)S_n}{m^n - 1}. \qquad (3)$$

$$a_1(S_n - a_1)^{n-1} = a_n(S_n - a_n)^{n-1}. \qquad (4)$$

$$a_n = a_1 m^{n-1}. \tag{5}$$

$$a_n = \frac{a_1 + S_n (m - 1)}{m}. \tag{6}$$

$$a_n = \frac{(m - 1) S_n m^{n-1}}{m^n - 1}. \tag{7}$$

$$a_n (S_n - a_n)^{n-1} = a_1 (S_n - a_1)^{n-1}. \tag{8}$$

$$S_n = \frac{a_1 (m^n - 1)}{m - 1}. \tag{9}$$

$$S_n = \frac{a_n m - a_1}{m - 1}. \tag{10}$$

$$S_n = \frac{a_n (m^n - 1)}{m^{n-1} (m - 1)}. \tag{11}$$

$$S_n = \frac{a_n^{\frac{n}{n-1}} - a_1^{\frac{n}{n-1}}}{a_n^{\frac{1}{n-1}} - a_1^{\frac{1}{n-1}}}. \tag{12}$$

$$m = \frac{S_n - a_1}{S_n - a_n}. \tag{13}$$

$$m = \left(\frac{a_n}{a_1}\right)^{\frac{1}{n-1}}. \tag{14}$$

$$m^n - \frac{S_n}{a_1} m = 1 - \frac{S_n}{a_1}. \tag{15}$$

$$m^n + \frac{S_n}{a_n - S_n} m^{n-1} = \frac{a_n}{a_n - S_n}. \tag{16}$$

$$n = \frac{\log. a_n - \log. a_1}{\log. m} + 1. \tag{17}$$

$$n = \frac{\log. a_n - \log. a_1}{\log. (S_n - a_1) - \log. (S_n - a_n)} + 1. \tag{18}$$

$$n = \frac{\log. [a_1 + (m - 1) S_n] - \log. a_1}{\log. m}. \tag{19}$$

$$n = \frac{\log. a_n - \log. [a_n m - (m - 1) S_n]}{\log. m} + 1. \tag{20}$$

HARMONIC SERIES.

514. An ***Harmonic Series*** is one, any *three consecutive terms* of which, form an harmonic proportion.

515. *Three terms* are in ***Harmonic Proportion*** when the first is to the third as the difference of the first and second is to the difference of the second and third.

Thus, 2, 3, and 6 are in harmonic proportion, since

$$2 : 6 \;=\; (3 - 2) : (6 - 3).$$

516. *Four terms* are in ***Harmonic Proportion*** when the first is to the fourth as the difference of the first and second is to the difference of the third and fourth.

Thus, 3, 4, 6, 9, are in harmonic proportion, since

$$3 : 9 \;=\; (4 - 3) : (9 - 6).$$

517. If a, b, and c are in harmonic proportion,

$$a : c \;=\; (a - b) : (b - c),$$

or $$ab - ac = ac - bc.$$

Dividing by abc, $\qquad \frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a}. \qquad$ That is,

The reciprocals of an harmonic series form an equidifferent series; and, conversely, it may be shown that *the reciprocals of an equidifferent series form an harmonic series.*

518. The reciprocals of the natural numbers,

$$1,\ \tfrac{1}{2},\ \tfrac{1}{3},\ \tfrac{1}{4},\ \tfrac{1}{5},\ \tfrac{1}{6},\ \text{etc.},$$

form the harmonic series to which the name was first applied, on account of the perfect harmony of musical strings of uniform size and tension, whose lengths are represented by the terms of this series.

519. To find the n^{th} term of an harmonic series:

RULE.—*Find the n^{th} term of the equidifferent series whose terms are the reciprocals of the terms of the given series and take its reciprocal.*

Thus, to find the 4th term of

10, 12, 15,

find the 4th term of $\frac{1}{10}$, $\frac{1}{12}$, $\frac{1}{15}$, which is $\frac{1}{20}$, and take its reciprocal 20. The series is then

10, 12, 15, 20.

520. To ***Interpolate Harmonic Means***, we have the

RULE.—*Form an equidifferent series from the reciprocals of the terms of the harmonic series, and interpolate the corresponding equidifferent means and take their reciprocals.*

Thus, to interpolate two harmonic means between 10 and 20, interpolate two equidifferent means between $\frac{1}{10}$ and $\frac{1}{20}$, and take the reciprocals.

EXAMPLES.

1. Find the 5th term of the harmonic series, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{6}$, etc.
2. Insert two harmonic means between $\frac{1}{6}$ and $\frac{1}{18}$.
3. Insert 3 harmonic means between 5 and 25.
4. Show that the equimultiple mean between two quantities is an equimultiple mean between their equidifferent and harmonic means.

DEVELOPMENT OF FORMULAS.

521. Develop from the solution of the following problems the formulas by which all similar problems may be solved.

1. What is the amount (a) of a note for p dollars, at compound interest t years, at r per cent?
2. What payment (p) made annually for t years will pay the sum s dollars, with compound interest at r per cent?
3. What sum (s) put at compound interest at $r\%$ will amount to a in t years?

4. What is the present worth (w) of a sum (s) due in t years, money being worth $r\%$ compound interest?

5. What is the present worth of a perpetual annuity (a), to commence in t years, at $r\%$ compound interest?

6. What is the present worth of an annuity (a), to commence in t years, and to continue t' years, allowing $r\%$ compound interest?

7. Find the amount of a note for \$500 on interest 3 years, at compound interest at 6%.

8. Find the present worth of \$1000 due in $3\frac{1}{2}$ years, allowing compound interest at 4%.

9. Find the sum that will amount to \$1000 in 5 years, on interest at 5%.

10. Find the present worth of an annuity to commence in 2 years and to continue 10 years.

11. Find the present worth of an annuity in arrears 3 years and to continue 7 years longer.

NOTE.—Use formula from Problem 6.

12. Find the present worth of a perpetual annuity of \$100 to commence in 5 years, at 6% compound interest.

13. The same as above, to commence now.

14. The same as above, to commence 5 years ago.

15. Find the time when an annual payment of \$100 should have begun to cancel at maturity a note for \$500 given Jan. 1, 1875, and due Jan. 1, 1880, with compound interest at 5%.

16. A man travels from a certain point northward 10 miles the first day, 9 miles the second, 8 miles the third, and so on, continuing the series by the same law. How far north will he be at the end of 5 days? How far at the end of 10 days? 11 days? 20 days? 22 days? 30 days? How far north will he travel the 15th day?

17. A man travels as above 10 miles the first day, and each succeeding day $\frac{9}{10}$ as far as the day before. How far north will he be in 5 days? In 10 days? How long would it require for him to travel 25 miles? How far would he travel if he should continue forever?

18. A ball falling from the height of 100 feet rebounds 50 feet. If it continue to rebound one half the distance it falls, how many times will it rebound? How far will it move before coming to rest?

522. The following *identical equation*

$$\frac{a}{m(m+p)(m+2p)\,..\,(m+rp)}=\frac{1}{rp}\left\{\frac{a}{m(m+p)\,..\,[m+(r-1)p]}-\frac{a}{(m+p)\,(m+2p)\,\,.\,.\,(m+rp)}\right\} \qquad \text{(E)}$$

furnishes a method for the summation of series whose terms are of the form of the first member.

The following examples will illustrate:

1. Find S_n and S_∞ of $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+$ etc.

SOLUTION.—Here

$a=1$; $m=1, 2, 3,$ etc.; $p=1$; and $r=1$.

Substituting in 2d member of (E),

$$S_n=\tfrac{1}{1}\left(\tfrac{1}{1}+\tfrac{1}{2}+\tfrac{1}{3}\,.\,.\,.+\frac{1}{n}-\tfrac{1}{2}-\tfrac{1}{3}\,.\,.\,.-\frac{1}{n}-\frac{1}{n+1}\right)$$

$$=1-\frac{1}{n+1}=\frac{n}{n+1}. \quad \textit{Ans.}$$

If $n=\infty$, $1-\frac{1}{n+1}=1-\frac{1}{\infty}=1$;

$$\therefore \quad S_\infty = 1.$$

2. Find S_∞ of $\frac{1}{3\cdot 8}+\frac{1}{6\cdot 12}+\frac{1}{9\cdot 16}+$ etc.

SOLUTION.—Multiplying the series by 12 gives

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\text{etc.},$$

which is the same as in Ex. 1.

$$\therefore \quad S_\infty = \tfrac{1}{12}.$$

3. Find S_∞ of $\dfrac{4}{1\cdot 2\cdot 3}+\dfrac{5}{2\cdot 3\cdot 4}+\dfrac{6}{3\cdot 4\cdot 5}+$ etc.

Solution. $a=4, 5, 6,$ etc.; $m=1, 2, 3,$ etc.; $p=1$; $r=2$.

Substituting in (E),

$$S_\infty=\tfrac{1}{2}\left(\frac{4}{1\cdot 2}+\frac{5}{2\cdot 3}+\frac{6}{3\cdot 4}+\frac{7}{4\cdot 5}+\text{etc.}-\frac{4}{2\cdot 3}-\frac{5}{3\cdot 4}-\frac{6}{4\cdot 5}-\text{etc.}\right)$$
$$=\tfrac{1}{2}\left(\frac{4}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\text{etc.}\right).$$

Applying (E) again to this series, beginning with the term $\dfrac{1}{2\cdot 3}$, we have,

$$a=1;\qquad m=2, 3, 4, \text{etc.};\qquad p=1;\qquad r=1.$$
$$S_\infty=\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}+\text{etc.}-\tfrac{1}{3}-\tfrac{1}{4}-\tfrac{1}{5}-\text{etc.}=\tfrac{1}{2}.$$
$$S=\tfrac{1}{2}\left(\frac{4}{1\cdot 2}+\frac{1}{2}\right)=\tfrac{5}{4}.\quad \textit{Ans.}$$

Find S_∞ of the following:

4. $\dfrac{1}{1\cdot 3\cdot 5}+\dfrac{4}{3\cdot 5\cdot 7}+\dfrac{7}{5\cdot 7\cdot 9}+$ etc.

5. $\dfrac{2}{3\cdot 5}-\dfrac{3}{5\cdot 7}+\dfrac{4}{7\cdot 9}-\dfrac{5}{9\cdot 11}+$ etc.

6. $\dfrac{1}{1\cdot 3}+\dfrac{1}{2\cdot 4}+\dfrac{1}{3\cdot 5}+$ etc.

7. $\dfrac{1}{1\cdot 3}-\dfrac{1}{2\cdot 4}+\dfrac{1}{3\cdot 5}-$ etc.

8. $\dfrac{4}{1\cdot 5}+\dfrac{4}{5\cdot 9}+\dfrac{4}{9\cdot 13}+$ etc.

9. $\dfrac{1}{1\cdot 2\cdot 3\cdot 4}+\dfrac{1}{2\cdot 3\cdot 4\cdot 5}+\dfrac{1}{3\cdot 4\cdot 5\cdot 6}+$ etc.

10. $\dfrac{1}{1\cdot 3\cdot 5\cdot 7}+\dfrac{1}{3\cdot 5\cdot 7\cdot 9}+\dfrac{1}{5\cdot 7\cdot 9\cdot 11}+$ etc.

11. $\dfrac{1^2}{1\cdot 2\cdot 3\cdot 4}+\dfrac{2^2}{2\cdot 3\cdot 4\cdot 5}+\dfrac{3^2}{3\cdot 4\cdot 5\cdot 6}+$ etc.

12. $\dfrac{1^2}{1\cdot 3\cdot 5\cdot 7}+\dfrac{2^2}{3\cdot 5\cdot 7\cdot 9}+\dfrac{3^2}{5\cdot 7\cdot 9\cdot 11}+$ etc.

REVERSION OF SERIES.

523. When we have $y =$ a series which is a function of x, finding x as a function of y is called ***Reverting the Series.***

Given, $\quad y = x + x^2 + x^3 + x^4 + x^5 + \text{etc.}, \qquad (1)$

to revert the series.

SOLUTION.—Assume

$$x = Ay + By^2 + Cy^3 + Dy^4 + \text{etc.} \qquad (2)$$

Substituting this value of x in (1),

$$\begin{array}{rl|l|l|l}
y = & Ay + B & y^2 + C & y^3 + D & y^4 + \text{etc.} \\
& + A^2 & \quad + 2AB & \quad + 2AC & \\
& & \quad + A^3 & \quad + B^2 & \\
& & & \quad + 3A^2B & \\
& & & \quad + A^4 &
\end{array}$$

Hence, Art. 419, $\qquad A = 1,$

$B + A^2 = 0; \qquad \therefore \ B = -1.$

$C + 2AB + A^3 = 0; \qquad \therefore \ C = 1.$

$D + 2AC + B^2 + 3A^2B + A^4 = 0; \qquad \therefore \ D = -1.$

The law of the series is evident and we may write

$$x = y - y^2 + y^3 - y^4 + y^5 - \text{etc.}$$

EXAMPLES.

Revert the following series:

1. $y = x + 2x^2 + 3x^3 + 4x^4 + \text{etc.}$
2. $y = 1 - x + x^2 - x^3 + x^4 - \text{etc.}$
3. $y = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \text{etc.}$
4. $y = 1 + x + 2x^2 + 3x^3 + \text{etc.}$

CHAPTER XX.

LOCI OF EQUATIONS.

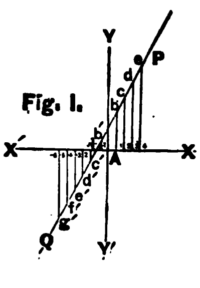
525. We have seen that an equation with two or more unknown quantities is *indeterminate;* that is, there are no definite values that can be assigned as the *only values* of these quantities. (Art. 241.)

For example, in the equation $x + y = 5$, we may have $x = 1$ and $y = 4$, $x = 2$ and $y = 3$, $x = 3$ and $y = 2$, and so on indefinitely.

And x not only may be any *integral* number between $+\infty$ and $-\infty$, but it may have any value, *integral, fractional,* or *incommensurable* between those limits. Hence, x and y are *variables,* and pass from one value to another by infinitesimal increments; that is, they pass through all intermediate values, as a point moving from one position to another along a line passes through all intermediate points. As the number of these points is infinite, so the number of different values of a variable is infinite. (Art. 390.)

526. The relation of an equation containing two variables, to a line considered as the path of a point moving in a plane surface, is more fully illustrated by the following method.

Fig. I.



527. Assume the two lines XX′ and YY′ (Fig. 1) at right angles to each other, intersecting at A. Take any equation with two variables, as

$$y = 2x + 3.$$

In this equation,

If $x = 0$, $y = 3$.	If $x = -1$, $y = 1$.
If $x = 1$, $y = 5$.	If $x = -2$, $y = -1$.
If $x = 2$, $y = 7$.	If $x = -3$, $y = -3$.
If $x = 3$, $y = 9$.	If $x = -4$, $y = -5$.
If $x = 4$, $y = 11$.	If $x = -5$, $y = -7$.
If $x = 5$, $y = 13$.	If $x = -6$, $y = -9$.

If, now, we adopt some convenient unit, and measure the positive values of x from A towards X (the negative values being of course measured in the opposite direction), we shall find on XX′ the several points 1, 2, 3, 4, etc., −1, −2, −3, −4, etc.

If from each of these points we erect a perpendicular equal to the corresponding value of y, we shall have the lines b1, c2, d3, and b′ (−1), c′ (−2), d′ (−3), etc.

If we have carefully drawn the lines to the proper measure, we may now place a ruler upon them, and connect all the points a, b, c, d, etc., by one straight line PQ.

We may also take fractional values for x and find corresponding values for y from the given equation, and all the lines representing the values of y will terminate upon this line PQ.

In like manner, *any point* on the line PQ represents a set of values for x and y by its distances from the lines YY′ and XX′.

Hence, the equation $y = 2x + 3$ is said to be the equation of the line PQ.

528. It is important for the student to become familiar with the definitions of the following terms used in discussions of this kind.

DEF. 1. ***The Axes***, or ***Axes of Reference*** are the assumed lines XX′ and YY′.

2. ***The Origin*** is their point of intersection A.

3. ***The Axis of Abscissas*** is the line XX′.

4. ***The Axis of Ordinates*** is the line YY'.

5. ***The Ordinate*** of a point is its distance from the axis of abscissas. Thus the ordinates of b, c, d, etc., Fig. I, are b1, c2, d3, etc.

6. ***The Foot of the Ordinate*** is the point where it meets the *axis of Abscissas.*

7. ***The Abscissa*** of a point is the distance from the origin to the foot of its ordinate, or the distance of the point from the *axis of ordinates.*

8. ***The Co-ordinates*** of a point are its *abscissa* and *ordinate.*

9. ***The Locus of an Equation*** is the line which the equation represents.

10. ***Constructing a Locus*** is drawing the line (as shown above) represented by an equation.

11. ***Abscissas*** are *positive* when measured to the *right,* and *negative* when measured to the *left* of the *axis of ordinates.*

12. ***Ordinates*** are *positive above* and *negative below* the *axis of abscissas.*

529. The four parts into which the plane is divided by the axes are called the *first, second, third,* and *fourth angle* respectively, beginning with the angle on the right of the axis of ordinates and above the axis of abscissas, and going round to the left.

Thus, YAX is the first, YAX' the second, Y'AX' the third, and Y'AX the fourth angle.

530. *Abscissas* are usually represented by the letter x, *ordinates* by y.

In the *first angle* x and y are both *positive.*
In the *second angle* x is *negative* and y *positive.*
In the *third angle* x and y are both *negative.*
In the *fourth angle* x is *positive* and y is *negative.*

531. Construct the loci of the following equations:

2. $y = 3x + 2.$ 5. $y = -3x + 2.$
3. $y = 2x - 1.$ 6. $y = -x.$
4. $y = 3x.$ 7. $y = 2.$

NOTE.—The last gives a line every point of which has its ordinate 2. Equations of the first degree always represent straight lines. They need not be in the form given above, but may for convenience be put in that form before constructing the loci.

Construct the following loci:

8. $\dfrac{x - y}{2} = \dfrac{x + y}{6}.$

9. $2x - 3y = 3.$

10. $\dfrac{x}{2y} - 5 = 0.$

532. When we make $x = 0$, we find the point where the line crosses the axis of ordinates, and when $y = 0$, the point where it crosses the axis of abscissas.

Thus in the equation

$$y = 2x + 3,$$

if $y = 0$, we have,

$$2x + 3 = 0,$$

and

$$x = -\tfrac{3}{2}.$$

This is the distance A $(-\frac{3}{2})$ (Fig. I), and it is the root of the equation $2x + 3 = 0$. Thus we see how we may construct the roots of equations having one unknown quantity.

11. Construct the root of $\dfrac{x - 4}{2} - \dfrac{x - 2}{3} = \dfrac{x - 8}{6}.$

SOLUTION.

Put the equation in the form,

$$\frac{x - 4}{2} - \frac{x - 2}{3} - \frac{x - 8}{6} = 0.$$

Make

$$y = \frac{x - 4}{2} - \frac{x - 2}{3} - \frac{x - 8}{6},$$

and construct the locus. The abscissa of the point where this locus cuts the axis of the abscissas will be the root required.

533. It appears from this, that *equations* of the *first degree* should give straight lines; for if the line could cut the axis of abscissas a second time, a *second root* would be found, which is not possible for such an equation. (Art. 232.)

12. Construct the locus

$$y = x^3 - 2x^2 - x + 2,$$

and find the real roots of

$$x^3 - 2x^2 - x + 2 = 0.$$

NOTE.—It is to be observed that the imaginary roots cannot be thus constructed.

Fig. II.

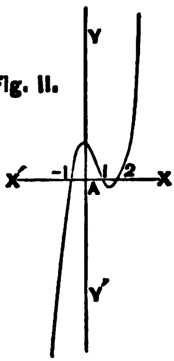


Fig. III.

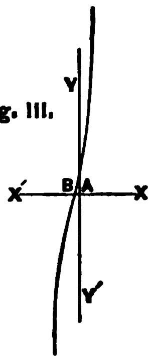
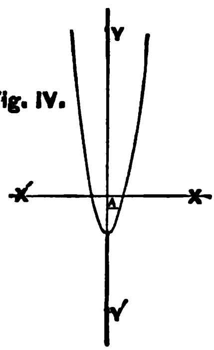


Fig. IV.



Assuming the axes XX′ and YY′ (Fig. II), and making

$x = 0,$	$y = 2;$
$x = \frac{1}{2},$	$y = \frac{9}{8};$
$x = 1,$	$y = 0;$
$x = -1,$	$y = 0;$
$x = 1\frac{1}{2},$	$y = -\frac{5}{8};$
$x = -1\frac{1}{2},$	$y = -4\frac{3}{8};$
$x = 2,$	$y = 0;$
$x = -2,$	$y = -12;$
$x = 3,$	$y = 8;$
$x = 4,$	$y = 30.$

The roots required are 2, 1, and -1, represented by the distances (A2), (A1), and [A (-1)].

This example illustrates the three roots of the equation of the third degree, showing how the line is cut by the axes of abscissas three times.

13. Construct the locus

$$x^3 + 3x^2 + 6x + 1 = y,$$

and find the value of x when $y = 0$. (See Fig. III.)

Fig. V.

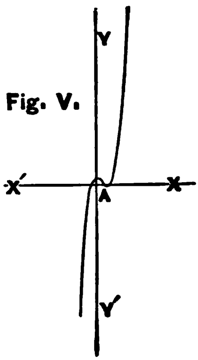


Fig. VI.

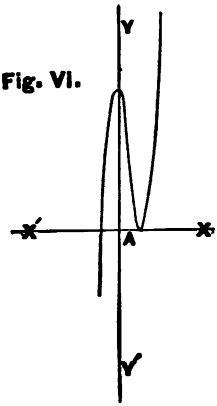
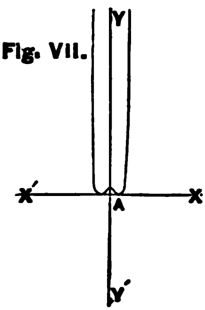


Fig. VII.



Making	$x = 0,$	$y = 1;$
	$x = 1,$	$y = 11;$
	$x = -1,$	$y = -3;$
	$x = 2,$	$y = 33;$
	$x = -2,$	$y = -7;$
	$x = -3,$	$y = -17.$

By constructing these points and others intermediate, by making $x = \frac{1}{2}$, $-\frac{1}{2}$, etc., and sketching in the curve to join the points, we may measure the distance AB, which will give us approximately the real root of the equation.

The construction also shows the other roots to be imaginary.

In like manner construct and find real roots of

14. $y = x^3 - 5$. (See Fig. IV.)
15. $y = x^3 - 2x^2 + 1$. (See Fig. V.)
16. $y = x^3 - 4x^2 - 3x + 18$. (See Fig. VI.)
17. $y = x^4 - 2x^2 + 1$. (See Fig. VII.)
18. $y = x^4 - 2x^2 - 1$. (See Fig. VIII.)
19. $y = x^4 - 2x^2 + 2$. (See Fig. IX.)
20. $y = x^5 - 3x^4 - 5x^3 + 15x^2 + 4x - 12$. (See Fig. X.)

Fig. VIII.

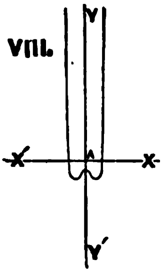


Fig. IX.

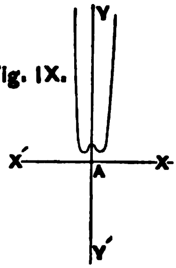
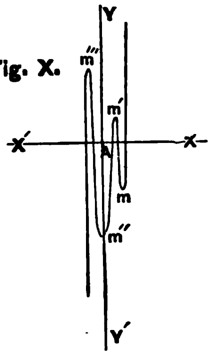


Fig. X.



NOTE.—The object of this chapter is to furnish illustrations of the principles to be developed relating to the general *Theory of Equations*, and not to give a practical method of finding the roots of equations.

CHAPTER XXI.

THEORY OF EQUATIONS.

534. When a problem involving but *one* unknown quantity produces an equation of the *first* or *second* degree, we can easily solve it in a general manner, by using general symbols, as letters, for the known quantities, thus obtaining a formula by which the unknown quantity may be found for all special cases of the problem, by mere arithmetical computation. Such formulas were obtained in Arts. 240, 340.

535. But when a problem gives rise to an equation of the *third* or *fourth degree*, the same may be done, but with much more difficulty, except in special cases. Indeed, so complicated is the reduction of the general equation of the fourth degree, that it is seldom employed in practice.

536. For equations above the *fourth degree*, no general method of reduction has yet been found. But when such equations arise, by putting for the known quantities *numerical values*, instead of *general symbols*, thus forming equations with numerical coefficients, called *numerical equations*, the *real roots* may be readily found.

By this method, the *arithmetical* part of the solution of the problem is performed before the *algebraic*, and the equation must be reduced for each special case of the problem.

537. The object of the present chapter is the discussion of the methods employed to find the *real roots* of *numerical equations of a higher degree* than can be readily reduced by the methods already given. Observe we do not say *methods of reducing higher equations*, for the *reduction* of an equation

implies the use of Axiom 1, Art. 38, in such manner as to bring out the unknown quantity as an *explicit function* of the known quantities. This is not done with higher equations; but by various devices depending on the ***General Theory of Equations,*** the *real roots* are discovered without a process of reduction.

538. To facilitate the discovery of the real roots of ***Higher Numerical Equations,*** they are reduced to the form

$$x^n + A_1x^{n-1} + A_2x^{n-2} \ldots\ldots A_{n-1}x + A_n = 0. \qquad (1)$$

in which A_1, A_2, etc., and n are integral and the exponents are all *positive.*

The reductions necessary to give this form to an equation may be any or all of the following:

1. To make the exponents *positive.*
2. To make the exponents *integral.*
3. To make the *coefficient* of x^n *unity.*
4. To make the *coefficients* A_1, A_2, etc., *integral.*

These transformations may be made as follows:

539. To Make the Exponents Positive.

RULE.—*Multiply the equation by x with a positive exponent equal numerically to the largest negative exponent in the equation.*

NOTE.—This will evidently accomplish the desired transformation, and will not affect the roots of the equation, since both members are equally affected. It is in fact only clearing the equation of fractions.

540. To Make the Exponents Integral.

RULE.—*Multiply each exponent by the least common multiple of the denominators of the fractional exponents.*

This will obviously make the *exponents* integral, but it will at the same time change the roots of the equation. It will

therefore be of no advantage to find the roots of the transformed equation, unless we can ascertain what function its roots will be of the primitive roots.

This we may easily do, for if m be the common multiple used, the transformation will be made by substituting y^m for x. Thus,

Let the original equation be

$$x^{\frac{1}{a}} + A_1x^{\frac{1}{b}} + A_2x^{\frac{1}{c}} + A_3 = 0.$$

Substituting $y^m = x$,

$$(y^m)^{\frac{1}{a}} + A_1(y^m)^{\frac{1}{b}} + A_2(y^m)^{\frac{1}{c}} + A_3 = 0.$$

Reducing,

$$y^{\frac{m}{a}} + A_1y^{\frac{m}{b}} + A_2y^{\frac{m}{c}} + A_3 = 0.$$

If m be made a multiple of a, b, and c, that is, abc, the equation becomes

$$y^{bc} + A_1y^{ac} + A_2y^{ab} + A_3 = 0,$$

in which each exponent has been multiplied by $m = abc$.

But since $y^m = x$, the values of y must be raised to the m^{th} power to give the values of x. Hence,

When the exponents of x in an equation are multiplied by m, the roots of the transformed equation raised to the m^{th} power will be the roots of the primitive equation.

541. To Make the Coefficient of x^n Unity.

RULE.—*Divide the equation by that coefficient.*

This evidently does not affect the roots.

542. To Make the Coefficients Integral without Changing the Coefficient of x^n.

RULE.—*Multiply the coefficient of x^{n-1} by k; that of x^{n-2} by k^2; that of x^{n-3} by k^3, and so on, to the coefficient of x^0, making k such a number as will render each coefficient integral.*

This rule is obtained by substituting $\frac{y}{k}$ for x, and clearing of fractions; thus,

Substituting $\frac{y}{k} = x$ in (1),

$$\frac{y^n}{k^n} + A_1 \frac{y^{n-1}}{k^{n-1}} + A_2 \frac{y^{n-2}}{k^{n-2}} \ldots\ldots + A_n = 0.$$

Clearing of fractions,

$$y^n + A_1 k y^{n-1} + A_2 k^2 y^{n-2} \ldots\ldots + A_n k^n = 0.$$

If, now, any of the coefficients, A_1, A_2, etc., are fractional, a value for k may be taken such as to remove the denominators. Since

$$\frac{y}{k} = x, \qquad y = kx.$$

That is, the roots have been *multiplied by k*, and when found must be *divided by k*, to give the *primitive roots.*

EXAMPLES.

Reduce the following equations to the form (1),

1. $2x^{\frac{5}{3}} - 5x^{\frac{4}{3}} + 7x^{\frac{1}{3}} - \frac{3}{4} = 0.$

SOLUTION.—Multiplying the exponents of x by 3, we have,

$$2y^5 - 5y^4 + 7y - \tfrac{3}{4} = 0,$$

in which $y^3 = x$. (Art. 540.) Dividing by 2 gives,

$$y^5 - \tfrac{5}{2}y^4 + \tfrac{7}{2}y - \tfrac{3}{8} = 0. \quad \text{(Art. 541.)}$$

Applying Rule, Art. 542,

$$z^5 - \frac{5k}{2}z^4 + \frac{7k^4}{2}z - \frac{3k^5}{8} = 0,$$

in which $z = ky$. Making $k = 2$, to cancel denominators,

$$z^5 - 5z^4 + 56z - 12 = 0.$$

in which $z = 2y$, and since $y^3 = x$,

$$z = 2x^{\frac{1}{3}}.$$

2. $3x^{\frac{7}{3}} - 5x^{\frac{5}{2}} + 8x - x^{\frac{1}{3}} - 1 = 0.$

3. $5x^{\frac{2}{3}} - 3x^{\frac{1}{4}} + 9x^{\frac{1}{3}} + 5 = 0.$

4. $\frac{3}{4}x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{2}{3}} + \frac{7}{8}x^{\frac{1}{3}} - \frac{1}{2} = 0.$

543. We have thus seen how any equation with a single unknown quantity having rational coefficients and exponents, may be reduced to the form (1) (in which the coefficient of x^n is unity, the other coefficients are integral, and the exponents integral and positive), without making any *unknown change* in the roots.

We now proceed to discuss this equation, for the purpose of discovering its *real roots*. For the sake of brevity we shall use for it the symbol

$$f(x) = 0.$$

I. DIVISIBILITY.

Theorem I.

544. *If $f(x)$ be divided by $x - a$, the remainder will be $f(a)$, that is, it will be what $f(x)$ becomes when a is substituted for x.*

Demonstration.—Let q be the quotient and r the remainder obtained by the division. Then

$$f(x) = (x - a)q + r,$$

an equation true independently of the value of x. It is therefore true when $x = a$. Substituting a for x, we have,

$$f(a) = r.$$

COR. 1.—*When a is a root of $f(x) = 0$, $f(x)$ is divisible by $x - a$, and not otherwise.*

For if a be a root, $f(a) = 0$. (Art. 217.) $\therefore r = 0$, and the division is complete. But if a be not a root, $f(a)$ is not 0, and r is not 0, and the division is not perfect.

COR. 2.—*If a be a root of $f(x) = 0$, a is a factor of the absolute term.*

For, it is evident that if $f(x)$ is divisible by $x - a$, the absolute term is divisible by a.

545. This theorem gives an easy method of determining whether any number be a *root* of an equation.

Try what factors of the absolute terms are roots of the following equations:

1. $x^3 - 8x^2 + 11x + 20 = 0.$
2. $x^5 - x^4 - 25x^3 + 85x^2 - 96x + 36 = 0.$
3. $x^5 - 1 = 0.$
4. $x^4 - 2x^3 + 8x - 16 = 0.$
5. $x^3 - 11x^2 + 43x - 65 = 0.$
6. $x^5 - 7x^3 + 2x + 15 = 0.$

In making these trials, use the method of synthetic division. (Art. 151.) Thus, to try -3 in Ex. 6, we write,

$$\begin{array}{l} 1 + 0 - 7 + 0 + \;\; 2 + 15 \,\underline{|\,-3} \\ \underline{\;\;\;\;\; - 3 + 9 - 6 + 18 - 60} \\ \;\;\;\;\; - 3 + 2 - 6 + 20 - 45, \text{ remainder.} \end{array}$$

The remainder is -45, and -3 is not a root.

546. We have also from this theorem a method of determining the value of a function when any number is substituted for x. Thus, when -3 is substituted for x in Ex. 6, the first member reduces to -45.

NOTE.—It should be observed that this theorem is applicable to all equations and not merely to those reduced to the form of (1).

Find the value of the first member of each of the preceding equations, when 1, 2, 3, -1, -2, and -3 are substituted for x.

II. NUMBER OF ROOTS.

THEOREM II.

547. *Every equation of an integral degree has the same number of roots as there are units in its degree.*

DEMONSTRATION.—To prove this theorem, it is necessary to *assume* that every such equation has at least *one root*, since the proof of this is not within the scope of elementary Algebra. But with this assumption we may easily prove the theorem, for, representing this root by a_1, and dividing the first member, when put in the form $f(x) = 0$, by $x - a_1$, we have

$$(x - a_1)q = f(x) = 0,$$

which is satisfied when $x - a_1 = 0$, or when $q = 0$. But q is a function of x one unit lower in degree than the original function, and $q = 0$ has by the assumption one root, which may be divided out as before, and this process may be continued so long as q is a function of x; hence the theorem is true, and $f(x) = 0$ has n roots and no more.

COR. 1.—*The first member of $f(x) = 0$ is the product of the n binomial factors,*

$$(x - a_1)(x - a_2)(x - a_3) \ldots . (x - a_n),$$

in which a_1, a_2, etc., are its roots.

NOTE.—It is not to be understood that a_1, a_2, etc., are all different numbers, for there is nothing in the proof of the theorem to forbid their being all equal.

III. FORMATION OF EQUATIONS.

COR. 2.—*An equation may be formed, having any roots whatever, by subtracting each root from x, and putting the product of the binomials thus formed equal to zero.* (Cor. 1.)

COR. 3.—*The coefficients of the several powers of x in the first member of $f(x) = 0$, will be as follows:*

Of x^{n-1}, the sum of the roots with their signs changed.

Of x^{n-2}, the sum of the products of the roots taken two and two. (Art. 383.)

Of x^{n-3}, the sum of the products of the roots, with their signs changed, taken three and three; and in general,

The coefficient of x^{n-p} will be the sum of the products of the roots, with their signs changed, taken p at a time.

This follows from the binomial formula, in which it is evident that the coefficients are formed in the same manner as in $f(x)$. Observe that in forming $f(x)$ by multiplying n binomial factors, the second terms of these factors are the roots *with their signs changed;* hence the products which form the several coefficients are all formed by the roots *with their signs changed*, but those products which have an *even* number of factors, as those taken, 2, 4, 6, etc, at a time, will have the *same sign* whether the signs of the roots be changed or not.

By this corollary an equation having any given roots may be formed by changing the signs of the roots, and forming the several coefficients in accordance with this law.

COR. 4. *The absolute term or coefficient of x^0 is the product of all the roots with their signs changed.*

NOTE.—Let the student notice how these principles applied to equations of the second degree give the same results as those already obtained in Chapter XI, Art. 329.

548. The truth of Theorem II is illustrated by the *loci of equations*, where the number of roots is represented by the number of times a straight line can cut the locus of the equation. This will be seen, by an examination of the loci already constructed, to be equal to the number of units in the degree.

Observe that it is not the number of times the *axis of abscissas* cuts the locus, for these are the real roots only, but *the number of times that any straight line can cut the locus* includes the *whole number* of roots, *real* and *imaginary.*

EXAMPLES.

Form the equations having the following roots, both by Cor. 2 and Cor. 3.

1. $5, \quad 3, \quad 3, \quad -1, \quad -3, \quad -5.$
2. $a, \quad b, \quad c, \quad d.$
3. $2, \quad 1+\sqrt{-3}, \quad 1-\sqrt{-3}.$
4. $2, \quad -2, \quad -3+\sqrt{-2}, \quad -3-\sqrt{-2}.$
5. $2\pm\sqrt{-5}, \quad 3\pm\sqrt{-7}, \quad -1\pm\sqrt{-1}.$

IV. FORMS OF ROOTS.

THEOREM III.

549. *The imaginary roots of $f(x) = 0$ will be found in conjugate pairs.*

That is, if $a + \sqrt{-b}$ be a root, $a - \sqrt{-b}$ will also be a root.

DEMONSTRATION.—Since the *sum* and the *product* of the roots are *both real*, there cannot be an *odd* number of *imaginary* roots, nor an *even* number, except when found in *conjugate pairs* whose *sum* and *product* are both real.

COR. 1. *$f(x)$ will have a real quadratic factor for each pair of imaginary roots.*

For $[x - (a + \sqrt{-b})] \times [x - (a - \sqrt{-b})] = (x - a)^2 + b.$

COR. 2. *$f(x)$ may be separated into real factors; of the first degree, corresponding to each real root; and of the second degree, corresponding to each pair of imaginary roots.*

COR. 3.—*The coefficients of $f(x)$ may be so changed as to give two equal or two unequal real roots in place of each pair of imaginary roots.*

For, the factors which produce the quadratic factor $(x-a)^2 + b$ are *imaginary* when b is *positive, real* and *equal* when b is *zero*, and *real* and *unequal* when b is negative. But a change of b from positive to negative through zero will produce no change in $f(x)$, except in its coefficients. Hence, the coefficients may be so changed as to require $b = 0$, or $b < 0$.

COR. 4.—*The product of the imaginary roots of $f(x) = 0$ is always positive.*

For, $(a + \sqrt{-b})(a - \sqrt{-b}) = a^2 + b$, which is always positive. Hence,

COR. 5.—*In an equation whose roots are all imaginary, the coefficient of x^0 will be positive.*

COR. 6.—*Every equation of an odd degree has at least one real root, opposite in sign to the absolute term, but an equation of an even degree may have all its roots imaginary.*

COR. 7.—*An equation of an even degree whose absolute term is negative, will have at least two real roots, one positive and one negative.*

This theorem and its corollaries are illustrated by the loci of equations.

For example, Figs. VII, VIII, and IX represent a locus of the 4th order in different positions with reference to the axis of abscissas. By reference to equations (16), (17), and (18) (Art. 534), which give those different positions, it will be observed that they differ only in the absolute term, an increase in this term carrying the locus upward and a decrease downward. This is evidently as it should be, for a change in the absolute term produces a corresponding change in each of the ordinates.

It appears therefore that a curve of an even degree, going as it does to infinity in only one direction and that upward, may be so placed, by giving a proper value to the absolute term of its equation, that the axis of abscissas will not cut it at all, as in Fig. IX, making all the values of x imaginary when $y = 0$.

Or, as in Fig. VIII, the axis may cut the curve twice only, giving two real and two imaginary roots. In passing from the position Fig. IX to Fig. VIII, it would pass the position Fig. VII, in which the roots are all real, two positive and two negative, the positive roots being equal, and also the negative.

By making the absolute term of the equation $\frac{1}{2}$, a position for the locus would be found in which the roots would all be real and unequal. It appears also from this illustration that when the equation is so changed as to drop out a real root, it must drop out two such roots; hence the number of imaginary roots will be even.

Fig. X shows a curve of the 5th order, or one whose equation is of the 5th degree. This curve goes to infinity both upward and downward, as do all curves whose equations are of an odd degree; see Figs. II, III, V, and VI. These curves cannot be moved up or down so far that they will not be cut at least once by the axis of abscissas. The roots of their equations, therefore, cannot all be imaginary.

In equation 19 (Fig. X), making the absolute term -25 will make all the roots but one imaginary, and that will be positive; making the absolute term 1 or any greater number will leave but one real root, and that will be negative.

THEOREM IV.

550. *If $f(x)$ has all its roots imaginary, it will have a positive value for every real value of x.*

DEMONSTRATION.—By the supposition, $f(x)$ is the product of factors of the second degree, each formed from a pair of imaginary roots; as,

$$(x - a + \sqrt{-b})(x - a - \sqrt{-b}) = (x - a)^2 + b.$$

Each of these factors is + for all values of x; hence their product is +. Notice how this is illustrated by Fig. IX. The curve being all above the axis of abscissas, when the roots are imaginary the ordinates are positive for all values of x.

COR.—*The sign of $f(x)$ for any real value of x will depend on the real roots.*

THEOREM V.

551. *The equation $f(x) = 0$ can have no fractional root.*

DEMONSTRATION.—Let $\frac{a}{b}$ represent a fraction reduced to its lowest terms, and suppose this fraction to be a root of $f(x) = 0$; then substituting, we have,

$$\frac{a^n}{b^n} + A_1\frac{a^{n-1}}{b^{n-1}} + A_2\frac{a^{n-2}}{b^{n-2}} + \ldots\ldots + A_n = 0.$$

Multiplying by b^{n-1},

$$\frac{a^n}{b} + A_1a^{n-1} + A_2a^{n-2}b + \ldots\ldots + A_nb^{n-1} = 0.$$

All the terms after the first in this equation are integral, and the first term is an irreducible fraction. But the sum of integers and an irreducible fraction cannot be zero. Hence the last equation is absurd, and $\frac{a}{b}$ is not a root of $f(x) = 0$.

COR.—*The real roots of $f(x) = 0$ will be either integral or incommensurable, since these are the only real quantities, except fractions.*

NOTE.—***Incommensurable Roots*** are such as cannot be measured with the unit of measure employed, and therefore require an infinite number of terms to express them. Such a quantity is $\sqrt{2}$, which gives rise to an endless decimal.

A ***Fraction*** is one or more of the *finite* parts of a unit, and can always be expressed by a *finite* numerator and denominator.

V. SIGNS OF THE ROOTS.

THEOREM VI.

552. *Changing the signs of the alternate terms of $f(x) = 0$ changes the signs of its roots.*

DEMONSTRATION.—By Theorem II, Cor. 3, changing the signs of the roots changes the signs of the alternate terms, beginning with the term containing x^{n-1}. But changing the signs of the alternate terms, beginning with x^n, gives the same equation by afterwards changing all the signs, which does not affect the roots.

Or the theorem may be proved as follows: If $+a$ and $-a$ be substituted for x in $f(x)$, the only difference in the result will be opposite signs for the terms containing the odd powers of x. Hence if a be a root of $f(x) = 0$, $-a$ will be a root of the same equation with the signs of the odd powers of x changed, or with the signs of the even powers changed, since this will give the same result.

THEOREM VII.

553. *In $f(x) = 0$, the number of positive roots cannot be greater than the number of variations of sign, nor the number of negative roots greater than the permanences.*

NOTE.—When two consecutive signs in an equation are *alike*, it is called a ***Permanence***; when *unlike*, a ***Variation***.

DEMONSTRATION.—Let the signs of $f(x)$, taken in their order, be $+ - + + - +$. If now a new positive root (a) be introduced, $f(x)$ will be multiplied by $x-a$. This multiplication will give, using only the signs,

$$\begin{array}{ccccccc}
+ & - & + & + & - & + & \\
+ & - & & & & & \\
\hline
+ & - & + & + & - & + & \\
 & - & + & - & - & + & - \\
\hline
+ & - & + & \pm & - & + & -
\end{array}$$

from which we see that using either sign, where there is an ambiguity we have added a variation. In like manner, multiplying by $+ +$ will add a permanence.

COR. 1.—*If the roots of an equation be all real, the number of positive roots will equal the number of variations, and the number of negative roots the number of permanences.*

For, in that case the whole number of roots will equal the whole number of permanences and variations together; hence by the theorem the corollary must be true.

COR. 2.—*If an equation be incomplete, and the signs before and after the missing term or terms be alike, the equation must have imaginary roots.*

For the intervening terms whose coefficients are zero may be either + or —, and we shall have, when there is one missing term, + ± + or — ± —, and in either case we may count either two permanences or two variations.

But the positive roots cannot be greater than the *least number* of variations, nor the negative roots greater than the *least number* of permanences; hence the two cannot be equal to the degree of the equation.

For example, in the equation

$$x^5 + 3x^3 - x + 7 = 0,$$

we shall have the signs

$$+ \quad \pm \quad + \quad \pm \quad - \quad +.$$

From which, counting as few variations as possible, we have 2; hence there cannot be more than 2 positive roots. Counting as few permanences as possible we have 1, and there cannot be more than 1 negative root. But there are 5 roots in all, hence there must be at least 2 imaginary roots.

COR. 3.—*If two or more consecutive terms are wanting, the equation will have imaginary roots whatever may be the signs of the preceding and following terms.*

How many positive and how many negative roots can the following equations have, and how many of their roots must be imaginary?

1. $x^7 - 5x^4 + 6x^2 - 5x + 1 = 0.$
2. $x^6 - 2x^2 + 5 = 0.$
3. $x^5 + x + 1 = 0.$
4. $x^5 - x^2 - 5 = 0.$

VI. LIMITS OF ROOTS.

554. ***A Superior Limit*** to the roots of an equation, is a number known to be larger than the largest root.

555. ***An Inferior Limit*** is a number known to be less than the least root.

It is sometimes convenient to find such limits which shall confine the search for roots within as narrow bounds as possible. This may be done by the following theorems:

THEOREM VIII.

556. *If, in $f(x) = 0$, two different numbers* (not roots) *be substituted for x, the signs of the result will be alike when there is an even, and unlike when there is an odd number of real roots situated between the numbers substituted.*

DEMONSTRATION.—Let

$$(x-a_1)(x-a_2)(x-a_3)\ .\ .\ .\ (x-a_{n-p}) \qquad (1)$$

(p being the number of imaginary roots) be the product of the factors of $f(x)$, which are formed from the real roots of $f(x) = 0$. The sign of this product for any value of x will be the same as of $f(x)$ for the same value of x; for the product of the imaginary factors is always $+$. (Art. 550).

Let it be supposed that the roots a_1, a_2, a_3, etc., are in the order of their magnitude, a_1 being greatest, a_2 the next, and so on.

If, now, a number (a') greater than a_1 be substituted for x, the factors of (1) will all be positive, and the result will be *positive;* but if a number (a'') less than a_1 and greater than a_2 be substituted for x, there will be *one negative factor*, and the rest will be positive. The result will therefore be *negative.*

In like manner if the number substituted be diminished till it is less than a_2, the result will change its sign again and become positive. Thus we see that for every root which is passed in diminishing the number substituted, the sign of the result changes, an even number of changes giving like signs and an odd number unlike. If there be two or more equal roots they will be passed at the same time, but this does not affect the truth of the theorem.

This theorem is also illustrated by the loci of equations. Referring to the preceding chapter it will be observed that the ordinate of any point of a locus is the value obtained by substituting the value of the abscissa for x in $f(x)$. By taking any two abscissas, it will be seen that the corresponding ordinates have opposite signs when the number of intermediate roots is odd, and the same sign when that number is even.

NOTE.—It must not be forgotten that *zero* is an *even* number.

COR.—*If a number less than the least root of $f(x) = 0$ be substituted for x, the result will be positive when the equation has an even number of real roots, and negative when an odd number.*

THEOREM IX.

557. *If A_m be the largest, and A_k the first negative coefficient of $f(x) = 0$, then $(A_m)^{\frac{1}{k}} + 1$ is a superior limit to its roots.*

DEMONSTRATION.—Any value of x which renders $f(x)$ positive, and of which it can be shown that all greater values will render it positive, is a superior limit to the roots. Suppose all the terms after the first negative term to be negative, and the coefficients of the negative terms each to be equal to A_m. As this will be the most unfavorable case possible, if we can show that when

$$x = \text{or} > (A_m)^{\frac{1}{k}} + 1, \tag{1}$$

then

$$x^n > A_m(x^{n-k} + x^{n-k-1} \ldots x + 1), \tag{2}$$

the theorem will be proved, for $f(x)$ will then be positive.

Subtracting 1 from each member of (1) and raising it to the k^{th} power,

$$(x - 1)^k = \text{or} > A_m,$$

or

$$(x - 1)^{k-1}(x - 1) = \text{or} > A_m;$$

$$\therefore \quad x^{k-1}(x - 1) > A_m,$$

and

$$1 > A_m \frac{x^{-(k-1)}}{x - 1}.$$

Multiplying by x^n,

$$x^n > A_m \frac{x^{n-k+1}}{x - 1},$$

or

$$x^n > A_m \frac{x^{n-k+1} - 1}{x - 1}.$$

Making the division,

$$x^n > A_m(x^{n-k} + x^{n-k-1} \ldots x + 1).$$

Hence, $(A_m)^{\frac{1}{k}} + 1$ *is a superior limit to the roots of* $f(x) = 0$, A_m *being the greatest negative coefficient and* k *the number of terms preceding the first negative coefficient.*

COR.—*If the signs of the alternate terms of an equation be changed, a superior limit to the roots of the resulting equation will, by changing its sign, become an inferior limit to the roots of the original equation.* (Theorem VI.)

VII. LIMITING EQUATION.

558. In the following discussions, $f'(x)$ represents the first differential coefficient of $f(x)$, and $\phi(x)$ the greatest common divisor of $f(x)$ and $f'(x)$.

THEOREM X.

559. *The real roots of* $f'(x) = 0$ *are situated between those of* $f(x) = 0$; *that is, if* a_1, a_2, a_3, *etc., are roots of* $f(x) = 0$, *then* $f'(x) = 0$ *has a root between* a_1 *and* a_2, *also between* a_2 *and* a_3, *and so on.*

DEMONSTRATION.—Suppose the roots of $f(x)$ to be real. Then we have,

$$\begin{aligned} f(x) &= (x-a_1)(x-a_2)(x-a_3)\ldots(x-a_n). \\ f'(x) &= (x-a_2)(x-a_3)(x-a_4)\ldots(x-a_n) + \\ &\quad (x-a_1)(x-a_3)(x-a_4)\ldots(x-a_n) + \\ &\quad (x-a_1)(x-a_2)(x-a_4)\ldots(x-a_n) + \\ &\quad \vdots \\ &\quad (x-a_1)(x-a_2)(x-a_3)\ldots(x-a_{n-1}), \end{aligned}$$

in which a_1, a_2, a_3, etc., are the roots of $f(x) = 0$, in the order of magnitude, a_1 being the largest. (Art. 414, Rule III.)

Any number equal to or greater than a_1, substituted for x in $f'(x)$, will render each factor of each term positive, and therefore the whole function positive; a_1 is therefore a *superior limit* to the roots of $f'(x) = 0$.

If a_2 be substituted, all the terms but the second will become zero, and that will have one negative factor and the function will therefore be negative.

So also a_3 substituted for x will make the function positive, and the successive substitutions of a_1, a_2, a_3, etc., will give alternately positive and negative values for $f'(x)$. Hence, (Theorem VIII) $f'(x) = 0$ has an odd number of roots between each two consecutive roots of $f(x) = 0$. This number cannot be less than one.

If $f(x) = 0$ has imaginary roots, the proof is not affected, since the sign of $f'(x)$ depends wholly upon its real roots.

560. The equation $f'(x) = 0$ is called the ***Limiting*** or ***Separating Equation***, on account of the situation of its roots between those of $f(x) = 0$.

Referring to Fig. X, Art. 533, we see that, when the equation of this locus is so changed in its absolute term as to give two equal roots, the axis of abscissas will pass through one of the points m, m′, m″, or m‴, and the equal roots will be the abscissa of one of these points. At the same time this abscissa will be a root of $f'(x) = 0$, since one of its roots is between the two equal roots of $f(x) = 0$. But the change in the absolute term of the equation of the locus, which caused the axis of abscissas to pass through m, makes no change in $f'(x)$. (Art. 413, Rule II.) Hence,

The roots of $f'(x) = 0$ *are the abscissas of the points* m, m′, *etc.*

VIII. EQUAL ROOTS.

Theorem XI.

561. *If* $f(x) = 0$ *has* m *roots each equal to* a, $f'(x) = 0$ *will have* $m - 1$ *such roots; and* $(x - a)^{m-1}$ *will be a common divisor of* $f(x)$ *and* $f'(x)$.

Demonstration.—This is evident from Theorem X, for if any two roots of $f(x) = 0$ become equal, the root of $f'(x) = 0$ between them must be the same, and if m roots of $f(x) = 0$ become equal, $m - 1$ roots of $f'(x) = 0$ must also be the same. This will obviously give $(x-a)^{m-1}$ as a common divisor of $f(x)$ and $f'(x)$. This common divisor will have other factors if $f(x)$ has other equal roots, and not otherwise.

This theorem may also be proved by reference to the form of $f(x)$ and $f'(x)$, under Theorem X.

A comparison of these functions shows: That if a_1, a_2, a_3, etc., are all unequal, there is no common divisor of the two, for each factor of $f(x)$ is wanting in some one term of $f'(x)$.

But if any two of the roots a_1, a_2, a_3, etc., are equal, making two of the factors of $f(x)$ equal, then that factor will be found in every term of $f'(x)$, and will therefore be a common divisor of the two functions. So if three of the factors of $f(x)$ are equal, that factor would be found twice in $f'(x)$, and therefore twice in the common divisor. Also if there be two sets of equal factors, the same would be true of each.

COR. 1.—*If $f(x) = 0$ has no equal roots, there is no common divisor of $f(x)$ and $f'(x)$.*

COR. 2.—*If $\phi(x) = 0$ has m roots each equal to a, $f'(x) = 0$ will have m, and $f(x) = 0$, $m + 1$ such roots.*

COR. 3.—*If $\phi(x) = 0$ has no equal roots, $f(x)$ is divisible by $\phi(x)$ twice, and each of the roots of $\phi(x) = 0$ is found twice as a root of $f(x) = 0$.*

562. This theorem furnishes the means of forming from any equation having *equal roots, two* or *more* equations whose roots shall be the roots of the given equation.

These equations are formed by putting $\phi(x)$ equal to zero, and also the quotient found by dividing $f(x)$ by $\phi(x)$.

In this way an equation of a higher degree may often be separated into factors and reduced. Take for illustration the following:

1. $$f(x) = x^5 - 15x^3 + 10x^2 + 60x - 72 = 0.$$

We find $$f'(x) = 5x^4 - 45x^2 + 20x + 60.$$

From this we may throw out the factor 5, giving

$$x^4 - 9x^2 + 4x + 12.$$

Finding the *g. c. d.* of these two functions, we have

$$\phi(x) = x^3 - x^2 - 8x + 12.$$

Testing this again for equal roots by trying for a common divisor of $\phi(x)$, and

$$\phi'(x) = 3x^2 - 2x - 8,$$

we find that divisor to be $x - 2$.

Putting $x - 2 = 0$,

we have $x = 2$,

one of the roots of $\phi(x) = 0$.

Dividing $\phi(x) = 0$ by $x - 2$, we have

$$x^2 + x - 6 = 0.$$

From this we get

$$x = -\tfrac{1}{2} \pm \sqrt{6 + \tfrac{1}{4}} = -\tfrac{1}{2} \pm \tfrac{5}{2} = 2 \text{ or } -3.$$

We have now the three roots of $\phi(x) = 0$, viz.:

$$2, \quad 2, \quad \text{and} \quad -3.$$

We might now divide $f(x) = 0$ by $\phi(x)$, and we should find

$$x^2 + x - 6 = 0,$$

which would give us the roots 2 and -3. But we need not make this division, since when we have the roots of $\phi(x) = 0$, viz.: 2, 2, and -3, we know that 2, which is twice a root of $\phi(x) = 0$, is three times a root of $f(x) = 0$; and -3, which is once a root of $\phi(x) = 0$, is twice a root of $f(x) = 0$. (Theorem XI, Cor. 2.) Hence the roots of $f(x) = 0$ are known to be 2, 2, 2, -3, and -3, as soon as we have the roots of $\phi(x) = 0$.

In like manner find the roots of

2. $x^8 - 6x^7 + 8x^6 + 18x^5 - 57x^4 + 36x^3 + 32x^2 - 48x + 16 = 0$.
3. $x^5 - x^4 + 4x^3 - 4x^2 + 4x - 4 = 0$.
4. $x^4 - 10x^3 + 24x^2 + 10x - 25 = 0$.
5. $x^5 + 2x^4 - 3x^3 - 3x^2 + 2x + 1 = 0$.

563. If an equation has roots numerically equal, but with opposite signs, changing the signs of the alternate terms will give an equation having the same numerically equal roots, but with its other signs different. Hence, the greatest common divisor of the two equations will have these equal roots and no others.

IX. COMMENSURABLE ROOTS.

564. The ***Commensurable Roots*** of $f(x) = 0$ are all *integral* (Art. 551, Theorem V), and may be found by the following theorem:

THEOREM XII.

565. *If a_r be a root of $f(x) = 0$, and A_n be divided by a_r and A_{n-1} added to the quotient, and if this sum be divided by a_r and A_{n-2} added to the quotient, and this process be continued till all the coefficients of $f(x)$ have been used, the result will be zero.*

DEMONSTRATION.—A careful consideration of the manner in which the coefficients are formed from the roots, will make the truth of the theorem plain. A^n is the product of the roots with their signs changed,

$\therefore -\frac{A_n}{a_r} =$ the product of all the roots, except a_r, *with their signs changed.*

A_{n-1} is the sum of the products of the roots, with their signs changed, taken $n - 1$ at a time. Each of the terms of A_{n-1} will therefore be divisible by a_r except one, and that term will equal $-\frac{A_n}{a_r}$. Therefore adding $\frac{A_n}{a_r}$ to A_{n-1}, cancels the only term which does not contain the factor a_r, and makes A_{n-1} divisible by a_r.

In like manner, A_{n-2} is the sum of the products of the roots, with their signs changed, taken $n-2$ at a time, and the terms not containing a_r will equal the last quotient obtained, with its sign changed. Hence, adding that quotient to A_{n-2} cancels the terms not containing the factor a_r, and renders A_{n-2} divisible by a_r.

In the same way each coefficient is made divisible by a_r. But A_1 is the sum of the roots, and when the terms not containing the factor a_r are cancelled, there will remain $-a_r$, which divided by a_r gives -1, and this added to 1, the coefficient of x^n, gives zero.

566. To find the integral roots of an equation by the application of this theorem, use the following

RULE.—I. *Write in a line the integral factors of the absolute term* (A_n).

II. *Divide A_n by each of these factors, and write each quotient below its divisor.*

III. *Add A_{n-1} to each quotient and write the sum below.*

IV. *Divide each sum by that factor of A_n which stands above it, and continue in like manner to add the successive coefficients and to divide, until the coefficients are all used.*

V. *If, in the course of the operation, the division by any one of the factors is imperfect, that factor is not a root and the work with it will cease.*

To illustrate the rule take Ex. 2. (Art. 562.) The operation will be as follows:

Divisors,	1,	2,	4,	8,	16,	−1,	−2,	−4,	−8,	−16,	Fac. of A_n.
	16,	8,	4,	2,	1,	−16,	−8,	−4,	−2,	−1,	Quotients.
Add	−48 = A_{n-1}.										
	−32,	−40,	−44,	−46,	−47,	−64,	−56,	−52,	−50,	−49,	Sums.
	−32,	−20,	−11,			64,	28,	13			Quotients.
Add	32 = A_{n-2}.										
	0,	12,	+21,			96,	60,	45,			Sums.
	0,	6,				−96,	−30,				Quotients.
Add	36 = A_{n-3}.										
	36,	42,				−60,	6,				Sums.
	36,	21,				60,	−3,				Quotients.
Add	−57 = A_{n-4}.										
	−21,	−36,				3,	−60,				Sums.
	−21,	−18,				−3,	30,				Quotients.
Add	18 = A_{n-5}.										
	−3,	0,				15,	48,				Sums.
	−3,	0,				−15,	−24,				Quotients.
Add	8 = A_{n-6}.										
	5,	8,				−7,	−16,				Sums.
	5,	4,				+7,	8,				Quotients.
Add	−6 = A_{n-7}.										
	−1,	−2,				+1,	2,				Sums.
	−1,	−1,				−1,	−1,				Quotients.

The work need be carried no farther to show that the *integral roots* are 1, 2, −1 and −2. This does not determine whether these roots are once or more than once roots of the equation, but the first member may now be divided by the factors $x-1$, $x-2$, $x+1$ and $x+2$, and the quotient put equal to zero will be an equation whose roots will be the other four roots of the primitive equation.

Making this division by the synthetic method as follows:

$$\begin{array}{rrrrrrrrr|r}
1 & -6 & +8 & +18 & -57 & +36 & +32 & -48 & +16 & 2 \\
 & 2 & -8 & +0 & +36 & -42 & -12 & +40 & -16 & \\
\hline
1 & -4 & +0 & +18 & -21 & -6 & +20 & -8 & +0 & -2 \\
 & -2 & +12 & -24 & +12 & +18 & -24 & +8 & & \\
\hline
1 & -6 & +12 & -6 & -9 & +12 & -4 & +0 & & 1 \\
 & +1 & -5 & +7 & +1 & -8 & +4 & & & \\
\hline
1 & -5 & +7 & +1 & -8 & +4 & +0 & & & -1 \\
 & -1 & +6 & -13 & +12 & -4 & & & & \\
\hline
1 & -6 & +13 & -12 & +4 & +0 & & & &
\end{array}$$

We have for the equation having the other four roots,

$$x^4 - 6x^3 + 13x^2 - 12x + 4 = 0.$$

Applying the rule for integral roots,

1,	2,	4,	−1,	−2,	−4,
4,	2,	1,	−4,	−2,	−1,
−12,					
−8,	−10,	−11,	−16,	−14,	−13,
−8,	−5,		16,	7,	
13,					
5,	8,		29,	20,	
5,	4,		−29,	−10,	
−6,					
−1,	−2,		−35,	−16,	
−1,	−1,		35,	8.	

We find 1 and 2 are roots of the equation.

Dividing out these roots gives

$$x^2 - 3x + 2 = 0,$$

whose roots are 1 and 2. Hence the roots of the primitive equation are

2, 2, 2, 1, 1, 1, −2, and −1.

567. Let the student now apply the principles already developed to the following

EXAMPLES.

Each of the following questions should be answered in reference to the equations below:

1. How many roots has the equation? (Theorem II.)

2. What is their product and what their sum? (Theorem II, Cor. 3.)

3. What numbers may be its integral roots? (Theorem I, Cor. 2.)

4. What limits to its roots can be fixed? (Theorem IX.)

5. Has it an even or an odd number of real roots? (Theorems II and III.)

6. Has it an even or an odd number of negative roots? (Art. 547, Cor. 4.)

7. Has it an even or an odd number of positive roots? (Art. 547, Cor. 4.)

8. How many positive and how many negative roots can the equation have? (Theorem VII.)

9. What are its commensurable roots? (Theorem XII.)

10. What is the value of the first member when 5 is substituted for x? (Theorem I.)

11. What equation has the same roots with opposite signs? (Theorem VI.)

12. What equation has its roots twice as large? (Art. 542.)

13. What equation has intermediate roots? (Theorem X.)

1. $x^6 + 45x^4 - 246x^2 + 200 = 0.$

2. $x^4 + 3x^3 - 2x - 156 = 0.$

3. $x^6 - 5x^4 + 3x^3 + 6x^2 - 297 = 0.$

4. $x^5 + 9x^4 + 7x^3 - 3x^2 - 4x + 10 = 0.$

5. $x^5 - 2x^4 + 3x^3 - 3x^2 + 2x - 1 = 0.$

6. $x^7 - 5x^6 + 3x^4 + 9x^3 + 12 = 0.$

Transform the following to the form (1) and give the changes produced in the roots by the transformation. (Arts. 538, 542.)

7. $x^{\frac{2}{3}} - 3x^{\frac{1}{2}} + 7x^{\frac{1}{3}} + \frac{1}{8}x^{\frac{1}{6}} - 1 = 0.$

8. $3x^{\frac{3}{5}} + 7x^{\frac{2}{5}} - 6x^{\frac{1}{5}} + 2x - 2 = 0.$

9. $2x^{\frac{1}{2}} + 3x^{\frac{1}{3}} + 4x^{\frac{1}{4}} - 2x + 2 = 0.$

10. $9x^{\frac{1}{2}} + 3x^{-\frac{1}{4}} - 2x^{-\frac{1}{3}} + 2 = 0.$

11. $\dfrac{x^2 - x}{2} - \dfrac{x^3 - x^2}{3} = \dfrac{x^3 - x}{x - 1}.$

Remove the equal roots from the following: (Theorem XI.)

12. $x^5 + x^4 - 4x^3 - 4x^2 + 4x + 4 = 0.$

13. $x^6 - 8x^5 + 26x^4 - 44x^3 + 41x^2 - 20x + 4 = 0.$

14. $x^6 - 3x^4 - 45x^2 - 81 = 0.$

Produce equations having the following roots:

1. $1, \ -2, \ 3.$

2. $3, \ 4, \ \sqrt{2}, \ -\sqrt{2}.$

3. $-5, \ \sqrt{-2}, \ -\sqrt{-2}.$

4. $-1 + \sqrt{-3}, \ -1 - \sqrt{-3}, \ \frac{1}{2}, \ 2.$

5. $1 \pm \sqrt{-3}, \ 3 \pm \sqrt{-2}.$

6. $2, \ 2, \ 3, \ 3, \ 1, \ 1.$

7. $\frac{1}{2}, \ \frac{2}{3}, \ \frac{7}{8}, \ \frac{5}{6}.$

8. $2, \ \frac{1}{2}, \ 1, \ 3, \ \frac{1}{3}.$

9. $\sqrt{-2}, \ -\sqrt{-2}, \ \sqrt{2}, \ -\sqrt{2}.$

10. $2, \ 2, \ 2, \ 2, \ 2.$

X. INCOMMENSURABLE ROOTS.

568. The process of finding the ***Incommensurable Roots*** of an equation depends on a theorem called, from its discoverer, "***Sturm's Theorem.***" *

This theorem assumes that the equation has no equal roots; but as we have already seen how the equal roots may be removed, we may prepare any equation for the application of the theorem.

569. Assuming, then, that $f(x)$ has no *equal roots,* and forming $f'(x)$, divide $f(x)$ by $f'(x)$, and represent the remainder *with its signs changed* by $f_{n-2}(x)$.

In like manner divide $f'(x)$ by $f_{n-2}(x)$, and represent the remainder with its signs changed by $f_{n-3}(x)$, and proceed in the same manner until a remainder, $f_0(x)$, is found.

NOTES.—1. The subscripts $n-2$, $n-3$, . . . 0, indicate the degree of the function. $f_0(x)$ is therefore independent of x, and such a remainder will be found; for, each division may be continued until the remainder is a unit lower in degree than the divisor, and the division will at no time be complete, since $f(x)$ has no equal roots. (Theorem XI.)

2. The use of these functions is such as not to forbid multiplying or dividing them by *any positive numerical factor.* They may therefore be simplified by rejecting all such factors, and to avoid fractions in dividing, any dividend or divisor may be multiplied or divided by any *positive number.*

570. The functions $f_{n-2}(x)$, $f_{n-3}(x)$, etc., are called the ***Sturmian Functions,*** and with $f(x)$ and $f'(x)$ constitute the functions to which *Sturm's theorem* relates.

The relations of these functions to each other is expressed in the following equations, in which Q_1, Q_2, etc., are the successive quotients.

---

* See page 307, Note 5.

$$f(x) = f'(x) \times Q_1 - f_{n-2}(x). \qquad (1)$$

$$f'(x) = f_{n-2}(x) \times Q_2 - f_{n-3}(x). \qquad (2)$$

$$f_{n-2}(x) = f_{n-3}(x) \times Q_3 - f_{n-4}(x). \qquad (3)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$f_2(x) = f_1(x) \times Q_{n-1} - f_0(x). \qquad (4)$$

571. ***Consecutive Functions*** are those adjacent, in the order $f(x)$, $f'(x)$, $f_{n-2}(x)$, $f_{n-3}(x) \dots f_0(x)$.

Theorem XIII.

572. *No two consecutive functions can become zero for the same value of x.*

Demonstration.—Suppose that $f_{n-2}(x)$ and $f_{n-3}(x)$ could become zero for the same value of x. Then by Equation (2), $f'(x) = 0$, and by Equation (1), $f(x) = 0$. But $f'(x)$ and $f(x)$ cannot become zero for the same value of x. (Art. 559.) $\therefore f_{n-2}(x)$ and $f_{n-3}(x)$ cannot be zero at the same time. The same may be proved of any two consecutive functions.

Cor.—*If $f'(x)$ or any one of the Sturmian functions reduces to zero for any value of x, the adjacent functions have opposite signs for the same value of x.*

If $f_{n-2}(x) = 0$, in Equation (2), we have

$$f'(x) = -f_{n-3}(x),$$

and in like manner for any other functions.

XI. STURM'S THEOREM.

573. *If in $f(x)$, $f'(x)$, $f_{n-2}(x)$, $f_{n-3}(x) \dots f_0(x)$, two different numbers be substituted for x, and the signs of the resulting values of the functions for each substitution be set separately in the order of the functions, the difference in the number of variations in the two cases will be equal to the number of real roots of $f(x) = 0$ situated between the numbers substituted.*

DEMONSTRATION.—1st. If the number substituted for x be supposed to change from one value to another, so as to pass through all intermediate values, the several functions will change their values in a similar manner, and whenever the value of any function passes from $+$ to $-$ or from $-$ to $+$, it will pass through zero.

2d. When any *intermediate* function becomes zero for any value of x, the adjacent functions have opposite signs for the same value of x. Hence a change of sign in any intermediate function (that is, in any function except the first and last) can have no effect on the number of variations. This is obvious, for the variations of $+ + -$ and $+ - -$ are the same.

3d. As none of the intermediate functions can affect the number of variations by any change that may occur in their signs, and as the last function, being independent of x, never changes its sign, any change in the number of variations must be produced by a change of sign in $f(x)$. But $f(x)$ will change its sign whenever the number substituted passes a root of $f(x) = 0$. If a number greater than a_1 be substituted, $f(x)$ and $f'(x)$ will both be positive and will form a permanence. If now this number be supposed to decrease when it passes a_1, $f(x)$ will change its sign, while $f'(x)$ will remain positive, since it has no root so great as a_1. This change of sign will make the first two signs $- +$, giving in place of a permanence a variation, and the number of variations will be increased by one. As the value of x continues to decrease, it will pass a root of $f'(x) = 0$ before it comes to a_2 (Th. X), and therefore $f'x$ will change its sign before $f(x)$ changes again. This will make the first two signs $- -$, but without affecting the variations. When therefore the value of x passes a_2, $f(x)$ will become positive and the first two signs will become $+ -$, adding another variation. In the same way a variation will be added whenever x passes a root of $f(x) = 0$. Hence, the theorem is proved.

574. ***Sturm's Theorem*** is applied to finding the situation of the incommensurable roots of an equation. It may also be used to find the commensurable roots, but the methods already given are sufficient to determine these.

A single example will illustrate its application.

1. Find the situation of the real roots of

$$x^3 - 3x^2 + 6x - 5 = 0.$$

SOLUTION.

$$f(x) = x^3 - 3x^2 + 6x - 5.$$
$$f'(x) \div 3 = x^2 - 2x + 2. \quad \text{(Art. 569, Note 2.)}$$
$$f_1(x) = -2x + 3.$$
$$f_0(x) = -1.$$

Substituting in these functions different values for x, we have the signs as follows:

		$f(x)$.	$f'(x)$.	$f_1(x)$.	$f_0(x)$.	
$+\infty$	gives	+	+	−	−	one variation.
0	"	−	+	+	−	two variations.
$-\infty$	"	−	+	+	−	two "

There is therefore one real root between $+\infty$ and 0, and no root between $-\infty$ and 0. That is, there is *one positive* and *no negative root*. Hence, two of the roots are imaginary.

To find the situation of the real root, we substitute as follows:

1	gives	−	+	+	−	two variations.
2	"	+	+	−	−	one variation.

The root is therefore between 1 and 2, or 1 + a decimal.

We may find more exactly the situation of this root by substituting 1.1, 1.2, 1.3, etc., as follows:

1.1	gives	−	+	+	−	two variations.
1.2	"	−	+	+	−	two "
1.3	"	−	+	+	−	two "
1.4	"	+	+	+	−	one variation.

Hence the root is between 1.3 and 1.4; that is, it is 1.3+.

In the same way other figures of the root could be found, but the substitutions would be tedious, and we shall hereafter show an easier method of carrying out the work after the first one or two figures have been found.

575. Find by *Sturm's Theorem* the first two figures of each incommensurable root of the following equations:

2. $x^3 + 3x^2 - 3x + 1 = 0$.
3. $x^5 - 5x^4 - 10x^3 + 10x - 1 = 0$.
4. $x^4 + 3x^3 - 6x + 2 = 0$.
5. $x^3 - 5x^2 + 7 = 0$.
6. $x^5 - x + 5 = 0$.
7. $x^7 - 7x^6 + 7x^5 - 14x^4 - 7x^3 + 7x^2 + 14x + 1 = 0$.
8. $x^3 - 6x^2 + 3x + 5 = 0$.
9. $x^3 + 6x^2 - 3x + 9 = 0$.
10. $x^3 + 5x^2 - 7x + 2 = 0$.

HORNER'S METHOD OF APPROXIMATION.*

576. *Horner's Method* of finding the successive figures of an *incommensurable root* is based on the following

PROBLEM.—*To find an equation whose roots shall be less than the roots of a given equation by a given number.*

SOLUTION.—Let x' be the number by which the roots of $f(x) = 0$ are to be diminished, and put y for the unknown quantity in the transformed equation. Then

$$y = x - x' \quad \text{or} \quad x = y + x'.$$

Substituting $y + x'$ for x, we have,

$$(y+x')^n + A_1(y+x')^{n-1} + A_2(y+x')^{n-2} \ldots .$$
$$A_{n-2}(y+x')^2 + A_{n-1}(y+x') + A_n = f(y+x') = 0. \qquad (1)$$

Developing the different powers of $(y+x')$ and collecting the terms containing like powers of y, and writing B_1, B_2, B_3, etc., for the coefficients, we have,

$$y^n + B_1y^{n-1} + B_2y^{n-2} \ldots . + B_{n-1}y + B_n = f(y+x') = 0. \qquad (2)$$

Substituting for y its value, $(x-x')$, this equation becomes,

$$(x-x')^n + B_1(x-x')^{n-1} + B_2(x-x')^{n-2} \ldots .$$
$$B_{n-1}(x-x') + B_n = f(x) = 0. \qquad (3)$$

If equation (3) be divided by $(x-x')$, the remainder will be B_n; and if the quotient be divided by $(x-x')$, the remainder will be B_{n-1}, and so on, by successive divisions by $(x-x')$, the remainders will be the successive coefficients of (2), beginning with the last. But the first member of equation (3) is equal to $f(x)$; hence if these successive divisions by $(x-x')$ be performed upon $f(x)$, the remainders will be the coefficients required.

In the following example the transformation is effected both by substitution and by the method of division:

1. Transform the equation $x^3 - 5x^2 + 3x - 7 = 0$, to another whose roots shall be 2 less than those of the given equation.

1st. *By Substitution.*

$$(y + 2)^3 - 5(y + 2)^2 + 3(y + 2) - 7 = 0.$$

---

* See page 307, Note 6.

Developing the equation,

$$\begin{array}{l|l|l|l}
y^3 + 6 & y^2 + 12 & y + \ \ 8 & = 0. \\
\quad - 5 & \quad - 20 & \quad - 20 & \\
 & \quad + \ \ 3 & \quad + \ \ 6 & \\
 & & \quad - \ \ 7 &
\end{array}$$

Or, $\qquad y^3 + y^2 - 5y - 13 = 0,$ *Ans.*

2d. *By Division.*

$$\begin{array}{rrrrrrrl}
1 & - & 5 & + & 3 & - & 7 & \underline{|2} \\
 & + & 2 & - & 6 & - & 6 & \\
\hline
 & - & 3 & - & 3 & ,- & \mathbf{13} & = B_3 \\
 & & 2 & - & 2 & & & \\
\hline
 & - & 1 & ,- & \mathbf{5} & = B_2 & & \\
 & & 2 & & & & & \\
\hline
 & ,+ & \mathbf{1} & = B_1 & & & &
\end{array}$$

$$y^3 + y^2 - 5y - 13 = 0, \quad \textit{Ans.}$$

NOTES.—1. The coefficients of the first terms of the successive quotients are omitted, as not essential to the work.

2. The coefficients of the transformed equation are printed in full-face type, and preceded by an inverted comma.

577. To apply this by Horner's method of approximating the incommensurable roots of an equation, take Ex. 1, Art. 574.

$$x^3 - 3x^2 + 6x - 5 = 0. \qquad (1)$$

We found that this equation has but one real root, and that this root is between 1.3 and 1.4.

To find other figures of the root, diminish the roots first by 1, as follows:

$$\begin{array}{rrrrrrrl}
1 & - & 3 & + & 6 & - & 5 & \underline{|1} \\
 & + & 1 & - & 2 & + & 4 & \\
\hline
 & - & 2 & + & 4 & ,- & \mathbf{1} & \\
 & + & 1 & - & 1 & & & \\
\hline
 & - & 1 & ,+ & \mathbf{3} & & & \\
 & + & 1 & & & & & \\
\hline
 & ,+ & \mathbf{0} & & & & &
\end{array}$$

This gives $\qquad y^3 + 0y^2 + 3y - 1 = 0, \qquad (2)$

an equation whose roots are 1 less than those of the primitive equation. Its real root therefore lies between .3 and .4.

Diminishing its roots by .3, as before,

$$\begin{array}{rrrrrrr|l}
1 & + & 0 & + & 3 & - & 1 & \underline{.3} \\
 & + & .3 & + & .09 & + & .927 & \\
\hline
 & + & .3 & + & 3.09 & - & \mathbf{.073} & \\
 & + & .3 & + & .18 & & & \\
\cline{2-5}
 & + & .6 & + & \mathbf{3.27} & & & \\
 & + & .3 & & & & & \\
\cline{2-3}
 & + & \mathbf{.9} & & & & &
\end{array}$$

and we have $\quad z^3 + .9z^2 + 3.27z - .073 = 0, \qquad (3)$

an equation whose roots are 1.3 less than those of the primitive equation. Its real root will therefore be the remaining figures of the real root of the primitive equation. Hence it is less than .1, its cube less than .001, and its square less than .01, and the omission of the first two terms will make but little difference in its root. We may therefore find the root approximately by using only the last two terms.

Thus, $\qquad 3.27z - .073 = 0,$

or $\qquad 3.27z = .073,$

and $\qquad z = \dfrac{.073}{3.27} = .02+.$

This gives, probably, the first figure of the root of (3), and the third figure of the root of (1).

Diminishing the roots of (3) by .02, we have,

$$\begin{array}{rrrrrrr|l}
1 & + & .9 & + & 3.27 & - & .073 & \underline{.02} \\
 & + & .02 & + & .0184 & + & .065768 & \\
\hline
 & + & .92 & + & 3.2884 & - & \mathbf{.007232} & \\
 & + & .02 & + & 0.0188 & & & \\
\cline{2-5}
 & + & .94 & + & \mathbf{3.3072} & & & \\
 & + & .02 & & & & & \\
\cline{2-3}
 & + & \mathbf{.96} & & & & &
\end{array}$$

The new equation is

$$w^3 + .96w^2 + 3.3072w - .007232 = 0. \qquad (4)$$

We know by this result that .02 is not greater than the first figure of the root of (3), for if it were, it would give a positive value on being substituted for x (Th. VIII), but the result of division shows that it gives a negative value. (Th. I.)

In the same manner that .02 was found from (3) we now find the next figure from (4) to be .002, and proceed to diminish the roots of (4) by .002, a process which may be continued indefinitely.

578. The operation of *diminishing the roots* by these successive figures may be written more compactly by omitting to re-write the coefficients of each new equation, and using them as they stand when first found.

We give Ex. 1 written in that manner, and designate the coefficients of each successive transformed equation by *half-parentheses* and *full-face type,* marking them by a subscript figure to indicate whether they belong to the first, second, or third transformed equation.

Thus, ${}_1(+3$, indicates a coefficient of the first transformed equation, and ${}_2(3.27$ a coefficient of the second transformed equation.

By a careful examination the student will see that the process is equivalent to the transformations above. Observe that when a decimal is to be added to an integer, it is *annexed* for greater economy of space.

1	− 3	+ 6	− 5	1.3221
	+ 1	− 2	+ 4	
	− 2	+ 4	${}_1$(− **1**	
	+ 1	− 1	.927	
	− 1	${}_1$(+ **3.09**	${}_2$(− **.073**	
	+ 1	+ .18	+ .065768	
	${}_1$(+ **0.3**	${}_2$(+ **3.27**	${}_3$(− **.007232**	
	+ .3	+ .0184	+ .006618248	
	+ .6	+ 3.2884	${}_4$(− **.000613752**	
	+ .3	+ .0188	+ .000331115	
	${}_2$(+ **.92**	${}_3$(+ **3.3072**	${}_5$(− **.000282637**	
	+ .02	+ .001924		
	+ .94	+ 3.309124		
	+ .02	+ .001928		
	${}_3$(+ **.962**	${}_4$(+ **3.311052**		
	+ .002	+ .00009661		
	+ .964	+ 3.31114861		
	+ .002			
	${}_4$(+ **.9661**			

Let the student extend the above, and find two or three more places of the root. He will observe that it is unnecessary to use more than two places of decimals in the first column, four in the second, and six in the third.

Let him also apply this method to equations in Art. 575, of which two figures of the roots have already been found.

In finding negative roots, change the signs of the alternate terms beginning with x^{n-1}, and thus change the signs of the roots. (Art. 552.)

The positive roots of the equation thus changed will, when their signs are changed, be the *negative* roots of the primitive equation.

2. Find the fifth root of 5.

The equation is $x^5 - 5 = 0$, which by Art. 296, 2°, has but one real root, and, without the application of Sturm's Theorem, we know the first figure to be 1. We may therefore proceed at once to apply Horner's Method, as follows:

$$
\begin{array}{rrrrrrl}
1 & +0 & +0 & +0 & +0 & -5 & \underline{|\,1} \\
 & +1 & +1 & +1 & +1 & +1 & \\
\cline{2-6}
 & +1 & +1 & +1 & +1 & {}_1(-4 & \\
 & +1 & +2 & +3 & +4 & & \\
\cline{2-5}
 & +2 & +3 & +4 & {}_1(+5 & & \\
 & +1 & +3 & +6 & & & \\
\cline{2-4}
 & +3 & +6 & {}_1(+10 & & & \\
 & +1 & +4 & & & & \\
\cline{2-3}
 & +4 & {}_1(+10 & & & & \\
 & +1 & & & & & \\
\cline{2-2}
 & {}_1(+5 & & & & &
\end{array}
$$

If we now divide 4 by 5 to find the next figure, we get .8, but this is evidently too large; for by Art. 556, the *first remainder* obtained in the next division must be *negative*. If, therefore, we use a figure so large as to give a positive result, we must reduce it. Let the student complete the work in this example.

3. Find the cube root of 1953125 by Horner's method.

4. Find the real roots of $x^4 - x^2 + 2x - 1 = 0$.

5. Find one root of $x^5 - 2x^4 - 3x^2 - 5x - 38756 = 0$.

6. Find a root of $x^3 - 2 = 0$.

7. Find the roots of $5x^3 - 3x - 1 = 0$.

579. To apply *Sturm's Theorem* and *Horner's Method of Approximation,* it is not necessary to reduce the equation to the form, Art. 538. To use synthetic division for the application of Horner's Method, we must however make the first coefficient unity. This will give for Example 7, the coefficients

$$1 + 0 - .6 - .2$$

8. Find the roots of $x^3 - 7x + 7 = 0$.

9. Find the roots of $x^3 - 5x - 5 = 0$.

10. Find the roots of $x^3 + 3x^2 + 3x + 5 = 0$.

580. The foregoing processes of finding the real roots of numerical equations may be summed up as follows:

1st. *Reduce the equation to the form*

$$x^n + A_1x^{n-1} + A_2x^{n-2} \dots + A_{n-1}x + A_n = 0,$$

by Arts. 539–542.

2d. *Try the factors of the last term by Art.* 566 *for commensurable roots.*

3d. *Divide out the roots thus found.* (Art. 544, Cor. 1.)

4th. *Apply Sturm's Theorem to the depressed equation, and if in the course of the process a common divisor between* $f(x)$ *and* $f'(x)$ *be found, form two equations by Theorem XI, and apply Sturm's Theorem to these equations to find the situation of the incommensurable roots.*

5th. *Find an approximation to each of the incommensurable roots by Horner's Method.*

581. The *imaginary roots* may be found, when there are only two, by *removing the other roots and reducing as a quadratic.*

582. When an equation has two nearly equal roots, it will be necessary to find, by Sturm's Theorem, a sufficient number of figures to separate the roots.

For example, in the following equation,

$$x^3 + 11x^2 - 102x + 181 = 0,$$

we find, by Sturm's Theorem, that the roots are all real, two being positive and one negative. We also find the positive roots situated between 3 and 4. Diminishing the roots by 3, gives the coefficients of the transformed equation,

$$1 + 20 - 9 + 1,$$

two of whose roots are between 0 and 1.

Applying Horner's Method, trying successively .1, .2, etc., we have,

$$\begin{array}{lllll|l} 1 & +\ 20 & -\ 9 & +\ 1 & & \underline{.1} \\ & +\ \ .1 & +\ 2.01 & -\ .699 & & \\ \hline & +\ 20.1 & -\ 6.99_6 & +\ .301 & & \end{array}$$

The remainder being +, both positive roots are either greater or less than .1. (Theorem VIII.)

$$\begin{array}{lllll|l} 1 & +\ 20 & -\ 9 & +\ 1 & & \underline{.2} \\ & +\ \ .2 & +\ 4.04 & -\ .992 & & \\ \hline & +\ 20.2 & -\ 4.96_6 & +\ .008 & & \end{array}$$

For the same reason these roots are both either greater or less than .2, and if we were to try .3, .4, etc., we should still find a positive remainder. Horner's Method, therefore, does not distinguish between these roots. But by further application of Sturm's Theorem, we find the positive roots of the primitive equation are situated between 3.2 and 3.3, and by Horner's Method we may then find the remaining figures.

Diminishing the roots by 3.2 gives the coefficients,

$$\begin{array}{lllll|l} 1 & +\ 20.6 & -\ .88 & +\ .008 & & \underline{.01} \\ & +\ \ .01 & +\ .2061 & -\ .006739 & & \\ \hline & +\ 20.61 & -\ .6739_6 & +\ .001261 & & \end{array}$$

and we see that both roots of the primitive equation are either greater or less than 3.21.

$$\begin{array}{lllll|l} 1 & +\ 20.6 & -\ .88 & +\ .008 & & \underline{.02} \\ & +\ \ .02 & +\ .4124 & -\ .009352 & & \\ \hline & +\ 20.62 & -\ .4676_6 & -\ .001352 & & \end{array}$$

We now know that one root is greater and one less than 3.22.

Let the student find each of these roots to 5 decimal places.

RECURRING EQUATIONS.

583. A ***Recurring Equation*** is one in which the coefficients of x^{n-r} and x^r are numerically equal, the corresponding coefficients all having either *like* or *unlike* signs.

584. A ***Reciprocal Equation*** is one whose roots are reciprocals of each other; that is, if a be a root, $\frac{1}{a}$ is also a root.

THEOREM XIV.

585. *A recurring equation is also Reciprocal.*

DEMONSTRATION.—The general recurring equation is

$$x^n + A_1x^{n-1} + A_2x^{n-2} \ldots \pm A_2x^2 \pm A_1x \pm 1 = 0. \qquad (1)$$

Substituting $\frac{1}{x}$ for x,

$$\frac{1}{x^n} + A_1\frac{1}{x^{n-1}} + A_2\frac{1}{x^{n-2}} \ldots \pm A_2\frac{1}{x^2} \pm A_1\frac{1}{x} \pm 1 = 0. \qquad (2)$$

Clearing of fractions,

$$1 + A_1x + A_2x^2 \ldots \pm A_2x^{n-2} \pm A_1x^{n-1} \pm x^n = 0. \qquad (3)$$

Which is the same as (1); hence the equation is satisfied when $\frac{1}{x}$ is put for x.

In equation (1), the double signs in the second half indicate that those terms are either all of the same sign as the corresponding terms of the first half, or all of contrary sign.

THEOREM XV.

586. *A Recurring Equation of an odd degree has* -1 *or* $+1$ *as a root, according as the corresponding terms have like or unlike signs.*

DEMONSTRATION.—Whenever the corresponding coefficients have unlike signs, the substitution of $+1$ for x will cause them to cancel each

other; and when they have like signs, -1 substituted for x will do the same; since one of these terms will be an even and the other an odd power of x.

COR.—*Such an equation may have its degree reduced one unit by dividing by* $x-1$ *or* $x+1$.

THEOREM XVI.

587. *A Recurring Equation of an even (the* $2n^{th}$*) degree, in which the coefficient of* x^n *is zero and the like coefficients have unlike signs, has both* $+1$ *and* -1 *as roots.*

DEMONSTRATION.—Represent the equation by

$$x^{2n}+A_1x^{2n-1}+A_2x^{2n-2}\ldots-A_2x^2-A_1x-1=0.$$

It is evident that both $+1$ and -1 will cause corresponding terms to cancel.

COR.—*Such an equation may be divided by* x^2-1, *and its degree thus reduced two units.*

THEOREM XVII.

588. *Every Recurring Equation of an even degree, whose corresponding terms have like signs, may be reduced to an equation of one-half that degree.*

DEMONSTRATION.—Let the equation be

$$x^{2n}+A_1x^{2n-1}+A_2x^{2n-2}\ldots+A_nx^n\ldots+A_2x^2+A_1x+1=0. \quad (1)$$

Dividing by x^n and uniting terms,

$$\left(x^n+\frac{1}{x^n}\right)+A_1\left(x^{n-1}+\frac{1}{x^{n-1}}\right)+A_2\left(x^{n-2}+\frac{1}{x^{n-2}}\right)\ldots A_n=0. \quad (2)$$

Make $$z=x+\frac{1}{x};$$

then, $$z^2-2=x^2+\frac{1}{x^2}.$$

$$z^3-3z=x^3+\frac{1}{x^3};$$

and in general $x^m + \frac{1}{x^m}$ may be expressed in terms of z, the highest power being z^m.

Substituting these values in (2), we have an equation of the n^{th} degree.

589. Recurring equations may often be reduced as quadratics, by reducing the degree in accordance with the preceding theorems. The following examples will make the student familiar with the principles.

EXAMPLES.

1. $x^5 - 3x^4 + 2x^3 - 2x^2 + 3x - 1 = 0.$

SOLUTION.—By Theorem XV, $+1$ is a root. Dividing out this root we have the equation,

$$x^4 - 2x^3 - 2x + 1 = 0.$$

Dividing by x^2,

$$x^2 + \frac{1}{x^2} - 2\left(x + \frac{1}{x}\right) = 0.$$

Substituting $z = x + \frac{1}{x}$,

$$z^2 - 2 - 2z = 0;$$

$$z = 1 \pm \sqrt{3};$$

$$x + \frac{1}{x} = 1 \pm \sqrt{3};$$

$$x^2 + 1 = (1 \pm \sqrt{3})x;$$

$$x = \frac{1 \pm \sqrt{3}}{2} \pm \tfrac{1}{2}\sqrt{\pm 2\sqrt{3}}. \qquad (a)$$

The roots written separately are: -1;

$$\frac{1 + \sqrt{3}}{2} + \tfrac{1}{2}\sqrt{2\sqrt{3}};$$

$$\frac{1 + \sqrt{3}}{2} - \tfrac{1}{2}\sqrt{2\sqrt{3}};$$

$$\frac{1 - \sqrt{3}}{2} + \tfrac{1}{2}\sqrt{-2\sqrt{3}};$$

$$\frac{1 - \sqrt{3}}{2} - \tfrac{1}{2}\sqrt{-2\sqrt{3}}.$$

NOTE.—Observe, that in equation (a) the double sign under the radical comes from the double sign in the numerator of the preceding term; hence, the upper signs of these must be taken together, and also the lower. But the double sign between the two terms has no dependence on the others, and may therefore be taken either way.

2. $x^3 - 2x^2 + 2x - 1 = 0.$

3. $x^4 - 3x^3 + 4x^2 - 3x + 1 = 0.$

4. $x^4 + 7x^3 - 7x - 1 = 0.$

5. $3x^5 - 2x^4 + 5x^3 - 5x^2 + 2x - 3 = 0.$

6. $x^6 - 3x^5 + 5x^4 - 5x^2 + 3x - 1 = 0.$

7. $x^4 - x^3 + x^2 - x + 1 = 0.$

8. $x^4 - x^3 + x - 1 = 0.$

9. $ax^4 - 2x^3 + 2x - a = 0.$

10. $5x^4 + 8x^3 + 9x^2 + 8x + 5 = 0.$

BINOMIAL EQUATIONS.

590. A ***Binomial Equation*** is one having but two terms, and is of the form

$$y^n \pm a^n = 0. \qquad (1)$$

By substituting ax for y, it takes the form

$$a^n x^n \pm a^n = 0, \quad \text{or} \quad x^n \pm 1 = 0. \qquad (2)$$

The roots of (2) multiplied by a will give the roots of (1).

591. The real roots of a binomial equation may be found by the usual method of evolution or by Horner's method of approximation.

592. The ***Imaginary Roots*** can be found by the method (Art. 303) which involves Trigonometry, or, if the degree of the equation be not too high, by solving the equation in form (2) as a recurring equation.

EXAMPLES.

1. Find the roots of $x^5 - 1 = 0$.

One root is 1, which being removed by division gives

$$x^4 + x^3 + x^2 + x + 1 = 0.$$

Dividing by x^2, $x^2 + \frac{1}{x^2} + x + \frac{1}{x} + 1 = 0.$

Putting $$z = x + \frac{1}{x},$$

$$z^2 + z - 1 = 0;$$

$$z = -\tfrac{1}{2} \pm \sqrt{1 + \tfrac{1}{4}} = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{5} = x + \frac{1}{x};$$

$$x^2 + (\tfrac{1}{2} \mp \tfrac{1}{2}\sqrt{5})x = -1;$$

$$x = -\tfrac{1}{4} \pm \tfrac{1}{4}\sqrt{5} \pm \sqrt{(\tfrac{1}{4} \mp \tfrac{1}{4}\sqrt{5})^2 - 1};$$

$$x = \tfrac{1}{4}(-1 \pm \sqrt{5} \pm \sqrt{-10 \mp 2\sqrt{5}}).$$

The roots of $x^5 - 1 = 0$ are, therefore, 1,

$$\tfrac{1}{4}(-1 + \sqrt{5} + \sqrt{-10 - 2\sqrt{5}});$$

$$\tfrac{1}{4}(-1 + \sqrt{5} - \sqrt{-10 - 2\sqrt{5}});$$

$$\tfrac{1}{4}(-1 - \sqrt{5} + \sqrt{-10 + 2\sqrt{5}});$$

$$\tfrac{1}{4}(-1 - \sqrt{5} - \sqrt{-10 + 2\sqrt{5}}).$$

The fifth roots of any other number will be these roots multiplied by the real 5th root of that number. For example, the roots of $x^5 - 32 = 0$ are the above, each multiplied by 2.

Find the roots of the following:

2. $x^3 \pm 1 = 0.$
3. $x^4 \pm 1 = 0.$
4. $x^6 \pm 1 = 0.$
5. $x^3 + 5 = 0.$
6. $x^5 + 7 = 0.$
7. $x^5 - 4 = 0.$
8. $x^7 + 5 = 0.$
9. $x^6 - 3 = 0.$

EXPONENTIAL EQUATIONS.

593. An ***Exponential Equation*** is an equation in which one or more of the exponents contains an unknown quantity.

1. Given $a^x = b$ to find x.

SOLUTION.—Taking the logarithms of both members,

$$x \log a = \log b. \quad \text{(Art. 460.)}$$

$$\therefore \qquad x = \frac{\log b}{\log a}.$$

2. Given $x^x = a$.

SOLUTION.—Taking the logarithms

$$x \log x = \log a.$$

Find by inspection from the table of logarithms and by trial, the value of x.

For example, let $\log a = 8$.

Then, $x \log x = 8$,

and by inspection, $x = 8.6$, nearly.

3. Given $10^x = 57$ to find x.
4. Given $27^x = 84$ to find x.
5. Given $x^{2x} = 13$ to find x.
6. Given $x^{x+2} = 5$ to find x.
7. In how many years will a sum of money double at compound interest, at 6%?
8. In how many years will a dollars amount to A at compound interest, at r%?
9. If a young man spends 25 cents a day for cigars, in how many years might he buy a farm worth \$5000 with the same money, by depositing it once a quarter, at 5% compound interest?

APPENDIX.

PROBABILITIES.

594. The ***Probability*** that any event will happen is the ratio of the *favorable* to the *whole number of chances.* This is on the supposition that the chances are all equally good.

595. Let it be known that a bag contains 10 balls, numbered from 1 to 10. If one ball be drawn, and there be no reason why it should be one number rather than another, we say there is one chance in ten that it will be a particular number, as No. 2 or No. 5. The chance then that No. 5 will be drawn is $\frac{1}{10}$, there being 10 events each equally likely to happen and only one of them being favorable.

If, of the 10 balls, 3 are white, 2 black and 5 red, the chance that a red ball will first be drawn is $\frac{5}{10}$ or $\frac{1}{2}$; that a black ball will be drawn is $\frac{2}{10}$ or $\frac{1}{5}$; and that a white ball will be drawn is $\frac{3}{10}$.

The chance that the first ball drawn will be either black or red, is $\frac{1}{5} + \frac{1}{2} = \frac{7}{10}$; and the chance that it will be either white, black or red, is $\frac{1}{5}+\frac{1}{2}+\frac{3}{10} = 1$. In this case the chance becomes a certainty, and is represented by 1.

596. If another bag contain 4 white, 3 black and 3 red balls, the chance of drawing the first time a red ball from this is $\frac{3}{10}$, while the chance of drawing a red from the first bag is $\frac{5}{10}$. The chance that both balls will be red is the product of the chances for each, or $\frac{5}{10} \times \frac{3}{10} = \frac{15}{100}$. This is evident, for any one of the balls in the first bag may be drawn with any one in the other, making the whole number of chances

the product of the whole number in one bag by the whole number in the other. Also, since each of the favorable chances in one case may happen with any one of the chances in the other, the product of the favorable chances in one case, multiplied by the favorable chances in the other case, will give the number of favorable chances when the two events are to occur together.

EXAMPLES.

1. What is the chance of throwing sixes, with two dice, at the first throw?

SOLUTION.—The chance that either of the dice will turn up a six is evidently $\frac{1}{6}$. Hence the chances of double sixes is $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$, *Ans.*

2. What is the chance of throwing sixes twice in succession?

SOLUTION.—The chance that two throws will give the same is $\frac{1}{36} \times \frac{1}{36} = \frac{1}{1296}$, *Ans.*

3. What is the chance that a throw of two dice will be greater than 6?

4. What is the chance that in drawing 100 numbers from a box, any three numbers will be drawn consecutively?

5. What is the chance that three points taken at random on a circle will be on the same semicircle?

6. What is the chance that two coppers tossed at random will both turn up heads? What is the chance that three coppers will do the same?

7. What is the chance that six coppers will turn up half heads and half tails?

8. What is the chance of drawing three white balls successively from a bag containing 10 white, 7 black and 5 red balls, the ball drawn being replaced before the next draw?

9. What would the chance be in Prob. 8, if the ball be not replaced?

10. What is the chance that one of each color will be drawn in the first three draws, not replacing balls drawn?

CARDAN'S FORMULA.

597. Cardan's formula for the reduction of cubic equations is obtained as follows:

The general form of a cubic equation is

$$x^3 + ax^2 + bx + c = 0. \qquad (1)$$

From this equation the second term may be made to disappear by the substitution of $y - \frac{a}{3}$ for x, giving

$$\left(y - \frac{a}{3}\right)^3 + a\left(y - \frac{a}{3}\right)^2 + b\left(y - \frac{a}{3}\right) + c = 0.$$

This diminishes each root of the equation by the quantity $-\frac{a}{3}$. (Art. 576.)

Developing and reducing,

$$y^3 + 0y^2 \begin{array}{r|} -\frac{1}{3}a^2 \\ +b \\ \end{array}\, y \begin{array}{r|} +\frac{2}{27}a^3 \\ -\frac{1}{3}ab \\ +c \end{array} = 0.$$

Substituting p for $b - \frac{1}{3}a^2$, and q for $\frac{2}{27}a^3 - \frac{1}{3}ab + c$ gives

$$y^3 + py + q = 0. \qquad (2)$$

Making $$y = z - \frac{p}{3z},$$

the equation becomes $z^6 + qz^3 - \frac{p^3}{27} = 0$,

which by Art. 329, (A), gives

$$z^3 = -\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}},$$

and $$z = \left(-\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{\frac{1}{3}},$$ therefore,

$$y\left(= z - \frac{p}{3z}\right) = \left(-\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{\frac{1}{3}} - \frac{p}{3\left(-\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{\frac{1}{3}}},$$

or, rationalizing the denominator and omitting the double signs, because they give no more values than single signs,

$$y = \left(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{\frac{1}{3}} + \left(-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{\frac{1}{3}}. \quad \text{(A)}$$

598. In equation (2), the coefficient of y^2 being o the sum of the roots is 0 (Art. 547, Cor. 3); hence, representing two of the roots by $m \pm \sqrt{n}$, the third root will be $-2m$. These roots give the equation

$$y^3 - (3m^2 + n)\,y + 2\,(m^3 - mn) = 0. \quad \text{(Art. 547, Cor. 2.)}$$

Equating with (2) gives the identical equation,

$$y^3 + py + q = y^3 - (3m^2 + n)\,y + 2\,(m^3 - mn);$$
$$\therefore \quad p = -(3m^2 + n);$$
$$q = 2\,(m^3 - mn).$$

Hence, $$\sqrt{\frac{q^2}{4} + \frac{p^3}{27}} = (m^2 - \tfrac{1}{9}n)\sqrt{-3n}. \quad (3)$$

The roots $-2m$ and $m \pm \sqrt{n}$ admit of three cases:

1. *When n is positive and the roots are all real.*
2. *When n is 0 and two of the roots are equal.*
3. *When n is negative and two of the roots are imaginary.*

The first case makes the second member of (3) imaginary, and therefore the terms of formula (A) become imaginary.

The second and third cases make (3) real, and formula (A) real. Hence,

When the roots of a cubic equation are all real and unequal, Cardan's formula has its terms imaginary.

This will occur when p [Eq. (2)] is negative and

$$\frac{q^2}{4} < -\frac{p^3}{27}.$$

599. The following examples illustrate the application of the formula to each of these cases:

1. Find the roots of $x^3 + 6x - 20 = 0$.

By the formula,

$$\begin{aligned}
x &= (10 + \sqrt{100 + 8})^{\frac{1}{3}} + (10 - \sqrt{100 + 8})^{\frac{1}{3}} \\
&= (10 + 6\sqrt{3})^{\frac{1}{3}} + (10 - 6\sqrt{3})^{\frac{1}{3}} \\
&= 1 + \sqrt{3} + 1 - \sqrt{3} \\
&= 2.
\end{aligned}$$

Removing this root from the equation,

$$\begin{array}{l|l}
1 + 0 + 6 - 20 & \underline{2} \\
\quad\;\; 2 + 4 + 20 & \\
\hline
\quad\;\; 2 + 10 &
\end{array}$$

The depressed equation is

$$x^2 + 2x + 10 = 0;$$

and $$x = -1 \pm \sqrt{-9} = -1 \pm 3\sqrt{-}.$$

The roots are 2, $-1 - 3$, and $-1 - 3$.

2. Find the roots of $x^3 - 3x - 2 = 0$.

By the formula,

$$\begin{aligned}
x &= (1 + \sqrt{1 - 1})^{\frac{1}{3}} + (1 - \sqrt{1 - 1})^{\frac{1}{3}} \\
&= 1 + 1 = 2.
\end{aligned}$$

Dividing by $x - 2$ and reducing the depressed equation, we find

$$x = -1 \pm 0,$$

3. Find the roots of $x^3 - 6x + 4 = 0$.

By the formula,

$$x = (-2 -^{\frac{1}{2}} 2)^{\frac{1}{3}} + (-2 -^{\frac{3}{2}} 2)^{\frac{1}{3}}. \tag{1}$$

This is called the "*Irreducible Case*," in which Cardan's formula fails to give the roots by ordinary methods of reduction. It occurs, as we have seen, when the roots of the equation are all real and unequal. It is, however, reducible by Arts. 297–305. We have, by Art. 303,

$$r(\cos\theta + \sqrt{-\sin^2\theta}) = \left(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right);$$

$$\therefore \qquad r\cos\theta = -\frac{q}{2}; \tag{2}$$

$$r\sqrt{-\sin^2\theta} = \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}. \tag{3}$$

Squaring (2) and (3) and subtracting,

$$r^2(\cos^2\theta + \sin^2\theta) = -\frac{p^3}{27};$$

or since $\cos^2\theta + \sin^2\theta = 1$,

$$r = \left(-\frac{p^3}{27}\right)^{\frac{1}{2}} = \left(-\frac{p}{3}\right)^{\frac{3}{2}};$$

and from (2),

$$\text{Cos}\,\theta = -\frac{q}{2} \div \left(-\frac{p}{3}\right)^{\frac{3}{2}} = -\frac{q}{2}\left(-\frac{3}{p}\right)^{\frac{3}{2}}.$$

This gives $\qquad r = (\tfrac{6}{3})^{\frac{3}{2}} = 2^{\frac{3}{2}} = 8^{\frac{1}{2}},$

and $\qquad \text{Cos}\,\theta = -2(\tfrac{1}{2})^{\frac{3}{2}} = -2(\tfrac{1}{8})^{\frac{1}{2}} = -\sqrt{\tfrac{1}{2}}.$

$$\therefore \qquad \theta = 135^\circ = -^{\frac{3}{4}}.$$

Hence, $\qquad -2 -^{\frac{1}{2}} 2 = -^{\frac{3}{4}} 8^{\frac{1}{2}},$

and $\qquad -2 -^{\frac{3}{2}} 2 = -^{\frac{5}{4}} 8^{\frac{1}{2}}.$

Substituting in (1),

$$x = (-^{\frac{3}{4}} 8^{\frac{1}{2}})^{\frac{1}{3}} + (-^{\frac{5}{4}} 8^{\frac{1}{2}})^{\frac{1}{3}}. \tag{4}$$

But $\qquad (-^{\frac{3}{4}} 8^{\frac{1}{2}})^{\frac{1}{3}} = -^{\frac{1}{4}} 2^{\frac{1}{2}}$, or $-^{\frac{11}{12}} 2^{\frac{1}{2}}$, or $-^{\frac{19}{12}} 2^{\frac{1}{2}}$

and $\qquad (-^{\frac{5}{4}} 8^{\frac{1}{2}})^{\frac{1}{3}} = -^{\frac{5}{12}} 2^{\frac{1}{2}}$, or $-^{\frac{13}{12}} 2^{\frac{1}{2}}$, or $-^{\frac{7}{4}} 2^{\frac{1}{2}}$. (Art. 302.)

Returning to the binomial form,

$$\left.\begin{aligned} -^{\frac{1}{4}} 2^{\frac{1}{2}} &= 2^{\frac{1}{2}}(\cos 45^\circ -^{\frac{1}{2}} \sin 45^\circ) \\ -^{\frac{11}{12}} 2^{\frac{1}{2}} &= 2^{\frac{1}{2}}(\cos 165^\circ -^{\frac{1}{2}} \sin 165^\circ) \\ -^{\frac{19}{12}} 2^{\frac{1}{2}} &= 2^{\frac{1}{2}}(\cos 285^\circ -^{\frac{1}{2}} \sin 285^\circ) \end{aligned}\right\} = (-^{\frac{3}{4}} 8^{\frac{1}{2}})^{\frac{1}{3}};$$

$$\left.\begin{aligned} -^{\frac{5}{12}} 2^{\frac{1}{2}} &= 2^{\frac{1}{2}}(\cos 75^\circ -^{\frac{1}{2}} \sin 75^\circ) \\ -^{\frac{13}{12}} 2^{\frac{1}{2}} &= 2^{\frac{1}{2}}(\cos 195^\circ -^{\frac{1}{2}} \sin 195^\circ) \\ -^{\frac{7}{4}} 2^{\frac{1}{2}} &= 2^{\frac{1}{2}}(\cos 315^\circ -^{\frac{1}{2}} \sin 315^\circ) \end{aligned}\right\} = (-^{\frac{5}{4}} 8^{\frac{1}{2}})^{\frac{1}{3}}.$$

Substituting the conjugate pairs in (4),

$$\begin{aligned} x &= 2^{\frac{1}{2}}(\cos 45^\circ -^{\frac{1}{2}} \sin 45^\circ) + 2^{\frac{1}{2}}(\cos 315^\circ -^{\frac{1}{2}} \sin 315^\circ) \\ &= 2.2^{\frac{1}{2}} \cos 45^\circ = 2; \end{aligned}$$

$$x = 2^{\frac{1}{2}}(\cos 165^\circ -^{\frac{1}{2}} \sin 165^\circ) + 2^{\frac{1}{2}}(\cos 195^\circ -^{\frac{1}{2}} \sin 195^\circ)$$
$$= 2.2^{\frac{1}{2}} \cos 165^\circ = -2.73205;$$

$$x = 2^{\frac{1}{2}}(\cos 75^\circ -^{\frac{1}{2}} \sin 75^\circ) + 2^{\frac{1}{2}}(\cos 285^\circ -^{\frac{1}{2}} \sin 285^\circ)$$
$$= 2.2^{\frac{1}{2}} \cos 75^\circ = .73205.$$

These results enable us to write more simple formulas for the roots of a cubic equation, when they are all real, as follows:

$$x = 2r \cos \tfrac{1}{3}\theta, \text{ or } 2r \cos \tfrac{1}{3}(\theta + 2\pi), \text{ or } 2r \cos \tfrac{1}{3}(\theta + 4\pi), \qquad \text{(B)}$$

in which the values of r and θ are as above.

In like manner find the roots of the following equations:

4. $x^3 - 3x^2 + 6x - 8 = 0.$
5. $x^3 + 5x - 6 = 0.$
6. $x^3 + 6x - 20 = 0.$
7. $x^3 - 8x + 35 = 0.$
8. $x^3 - 15x - 2 = 0.$
9. $x^3 - 9x + 5 = 0.$
10. $x^3 - 3x - 1 = 0.$
11. $x^3 - 7x + 5 = 0.$
12. $x^3 - 6x + 3 = 0.$

DESCARTES'S FORMULA.*

For the Reduction of ***Biquadratics.***

600. Let it be required to reduce the general biquadratic,

$$x^4 + A_1x^3 + A_2x^2 + A_3x + A_4 = 0. \qquad (1)$$

Transforming (1) to remove the second term (Art. 597), gives an equation of the form,

$$y^4 + a_1y^2 + a_2y + a_3 = 0. \qquad (2)$$

Assume,

$$y^4 + a_1y^2 + a_2y + a_3 = (y^2 + my + n)(y^2 + py + q). \quad (3)$$

Developing and collecting terms,

$$y^4 + a_1y^2 + a_2y + a_3 = y^4 + \left.\begin{matrix} m \\ +p \\ \end{matrix}\right| y^3 + \left.\begin{matrix} n \\ +mp \\ +q \end{matrix}\right| y^2 + \left.\begin{matrix} np \\ +mq \\ \end{matrix}\right| y + nq.$$

* See page 308, Note 8.

This equation being identical, gives by Art. 419, Cor.,

$$m + p = 0; \tag{4}$$

$$n + mp + q = a_1; \tag{5}$$

$$np + mq = a_2; \tag{6}$$

$$nq = a_3. \tag{7}$$

From (4), $$p = -m.$$

Substituting in (5) and (6),

$$n - m^2 + q = a_1;$$

$$m(q - n) = a_2;$$

from which we find,

$$n = \frac{1}{2}\left(m^2 - \frac{a_2}{m} + a_1\right);$$

$$q = \frac{1}{2}\left(m^2 + \frac{a_2}{m} + a_1\right).$$

Substituting in (7) we have,

$$m^6 + 2a_1m^4 + (a_1^2 - 4a_3)\,m^2 - a_2^2 = 0. \tag{8}$$

Substituting for m,

$$\sqrt{m_1 - \tfrac{2}{3}a_1},$$

gives a cubic equation from which m_1 may be found by Cardan's formula, thus giving values for m, n, p, and q.

These values substituted in the second member of (3), and each quadratic factor being separately made equal to zero, enables us to find y and then x.

The student may apply this process to the following equation:

$$x^4 - 4x^2 - 8x + 11 = 0.$$

CONTINUED FRACTIONS.*

601. A ***Continued Fraction*** is one whose numerator is a whole number, and whose denominator is a whole number plus a fraction, which also has a whole number for its numerator, and for its denominator a whole number plus a fraction; and so on.

We shall consider only those in which each of the numerators is unity, and the partial denominators (a, below) are all positive. Thus,

$$\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\text{etc.}}}} \qquad (1)$$

$$\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\text{etc.}}}} \qquad (2)$$

are continued fractions.

602. The integral parts of the denominators are sometimes called *partial* denominators, or *partial quotients;* and the fractions, $\frac{1}{2}$, $\frac{1}{3}$, etc., $\frac{1}{a_1}$, $\frac{1}{a_2}$, etc., are called *partial,* or *integral* fractions.

603. If, in (1) above, we neglect all but the *first* partial fraction, the denominator 2 will be *less* than the true denominator; and, of course, $\frac{1}{2}$ is *greater* than the true value of the continued fraction.

Again, suppose we neglect all but *two* partial fractions. Then the partial denominator 3, being *too small,* the partial

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* This article is taken substantially from the excellent work of Professor Stephen Chase, late of Dartmouth College.

fraction, $\frac{1}{3}$, is *too great;* and, consequently, $2\frac{1}{3}$ being greater than the true denominator, the fraction,

$$\frac{1}{2+\frac{1}{3}} = \frac{1}{\frac{7}{3}} = \frac{3}{7},$$

will be *less* than the true value of the continued fraction.

604. Similar reasoning will, evidently, hold in respect to *any number* of terms, and will apply equally to the general form (2), as to the particular example we have considered.

Hence,

***If we include in the reduction an* ODD *number of partial fractions, the result will be too* GREAT*; if an* EVEN *number, the result will be too* SMALL.**

605. The fractions,

$$\frac{1}{a_1}, \qquad \frac{1}{a_1+\frac{1}{a_2}}, \qquad \frac{1}{a_1+\frac{1}{a_2+\frac{1}{a_3}}}, \qquad \text{etc.},$$

are *approximate values* of the given fraction, and are sometimes called *approximating* or *converging* fractions, or simply ***Convergents.***

It is evident that the *true* value of the continued fraction, lying between two successive *approximate* values, *differs from either of them less than they differ from each other.*

606. We have

$$\frac{1}{a_1} = \frac{1}{a_1}, \qquad \text{1st approx. value.}$$

$$\frac{1}{a_1+\frac{1}{a_2}} = \frac{a_2}{a_1a_2+1}, \qquad \text{2d} \quad \text{“} \quad \text{“}$$

$$\frac{1}{a_1+\frac{1}{a_2+\frac{1}{a_3}}} = \frac{a_2+\frac{1}{a_3}}{a_1\left(a_2+\frac{1}{a_3}\right)+1} = \frac{a_2a_3+1}{(a_1a_2+1)a_3+a_1},$$

. . . . 3d approximate value.

We shall evidently find the *fourth* approximate value, or convergent, by substituting, in the *third*, $a_3 + \frac{1}{a_4}$ for a_3.

Thus,

$$\frac{(a_2a_3 + 1)\,a_4 + a_2}{[(a_1a_2 + 1)\,a_3 + a_1]\,a_4 + a_1a_2 + 1}$$

is the *fourth* convergent.

We obviously find the ***numerator and denominator of the third convergent by multiplying those of the second by the third partial denominator, and adding those of the first convergent.***

We find, in like manner, the *fourth convergent* from the terms of the *second and third.*

607. To show the generality of this law, let it be admitted to hold good as far as the n^{th} convergent (*i. e.*, the convergent corresponding to a_n).

Let also $\frac{L}{L'}$, $\frac{M}{M'}$, $\frac{N}{N'}$, and $\frac{P}{P'}$ be the convergents corresponding to a_{n-2}, a_{n-1}, a_n, and a_{n+1}.

Then, since the n^{th} convergent is formed according to the above law, we shall have

$$\frac{N}{N'} = \frac{Ma_n + L}{M'a_n + L'}. \tag{3}$$

If now we substitute in $\frac{N}{N'}$, $a_n + \frac{1}{a_{n+1}}$ for a_n, we shall obviously find $\frac{P}{P'}$. Thus,

$$\frac{P}{P'} = \frac{M\left(a_n + \frac{1}{a_{n+1}}\right) + L}{M'\left(a_n + \frac{1}{a_{n+1}}\right) + L'} = \frac{(Ma_n + L)\,a_{n+1} + M}{(M'a_n + L')\,a_{n+1} + M'};$$

or, from (3),

$$\frac{P}{P'} = \frac{Na_{n+1} + M}{N'a_{n+1} + M'}. \tag{4}$$

Consequently, if the law holds good for n convergents, it will for $n + 1$. Hence,

608. To Find the Numerator and Denominator of any Convergent after the Second, as the $(n + 1)^{th}$, we have the following

RULE.—*Multiply the numerator and denominator of the n^{th} convergent by the $(n + 1)^{th}$ partial denominator, and add to the products, respectively, the numerator and denominator of the $(n - 1)^{th}$ convergent.*

609. The numerator and denominator of any convergent must be respectively *greater* than those of the preceding; each numerator and each denominator being at least equal to the sum of the two next preceding.

610. Moreover, each convergent is found by substituting in the preceding, for the last *partial* denominator, an expression known to approach more nearly to the *true* denominator.

Hence, evidently, each convergent approximates more closely than the preceding to the true value of the continued fraction.

1. Find the successive convergents of the continued fraction,

$$\cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{87}}}}}$$

Ans. $\frac{1}{2}$, $\frac{1}{3}$, $\frac{3}{8}$, $\frac{4}{11}$, and $\frac{351}{965}$.

The first four convergents are *approximate* values of the continued fraction; the last, $\frac{351}{965}$, is the *true* value.

611. A continued fraction is sometimes ***mixed,*** or made up of a whole number and a fraction. Thus,

$$3 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{5 + \text{etc.}}}} \qquad (5)$$

In such cases, the integral part may be reserved and added to the convergents; or it may be taken, with 1 as a denominator, for the first convergent.

612. Thus, in the above example, we shall have the convergents,

$$3\tfrac{1}{2},\ 3\tfrac{3}{7},\ 3\tfrac{16}{37}, \quad \text{or} \quad \tfrac{3}{1},\ \tfrac{7}{2},\ \tfrac{24}{7},\ \tfrac{127}{37}.$$

This form,
$$a + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \text{etc.},}} \tag{6}$$

is sometimes assumed as the general form of a continued fraction; the place of the integral part, when it is wanting, being filled with 0.

In that case, the *first* convergent is evidently too *small*, the *second* too *great*, and so on, those of an *odd* order being too *small*, and those of an *even* order too *great*. (Art. 604.)

NOTE.—If the integral part be *zero*, the first convergent will of course be $\frac{0}{1}$.

613. If the second convergent of Art. 606 be subtracted from the first, the remainder is *unity* divided by the product of the denominators. If the *third* be subtracted from the *second*, the remainder is *minus unity* divided by the product of the denominators.

Suppose it has been proved that this law extends to $n - 1$ convergents; that is,

$$\frac{L}{L'} - \frac{M}{M'} = \frac{LM' - L'M}{L'M'} = \frac{\pm 1}{L'M'}. \tag{7}$$

Then
$$\frac{M}{M'} - \frac{N}{N'} = \frac{M}{M'} - \frac{Ma_n + L}{M'a_n + L'}$$

$$= \frac{L'M - LM'}{M'N'} = -\frac{LM' - L'M}{M'N'}, \tag{8}$$

the numerator of which is the same as that of (7), with a contrary sign. Hence, the principle proved in regard to the first three convergents, applies equally to the whole series. For,

If each convergent be subtracted from that which next precedes, the numerator of the difference will be ± 1, *and the denominator will be the product of the denominators of the two convergents.*

614. Again, the *true value* of the continued fraction lies between any two successive convergents, and differs from either of them less than they differ from each other. (Art. 605.)

That is, the convergent $\frac{M}{M'}$, differs from the true value of the continued fraction by less than $\frac{1}{M'N'}$.

$$\text{But (Art. 608),} \qquad M' < N',$$

$$\text{and} \quad \therefore \quad M'^2 < M'N'.$$

$$\therefore \quad \frac{1}{M'N'} < \frac{1}{M'^2}. \quad \text{Hence,}$$

COR. 1.—*The error in taking any convergent whatever for the true value of the continued fraction is numerically less than unity divided by the square of the denominator of that convergent.*

615. The denominator of each convergent is greater than the next preceding by some whole number. (Art. 609.)

Hence, if the fraction be *infinite*, we may find a convergent whose denominator shall be *greater* than any given quantity; and, consequently,

COR. 2.—*We may find a convergent which shall differ from the true value of the continued fraction by less than any given quantity.*

616. Suppose that M and M' have a common divisor, D.

Then D will of course divide $L'M$ and LM', multiples of M and M', and consequently the difference of those multiples, $LM' - L'M = \pm 1$.

Therefore D must divide ± 1, which has no integral divisor but unity.

$$\therefore \quad D = 1. \quad \text{Hence,}$$

COR. 3.—*Every convergent is in its lowest terms.*

617. One of the most obvious uses of continued fractions is to express approximately, in small numbers, fractions whose terms are large. Thus,

1. $$\frac{17}{59} = \frac{1}{\frac{59}{17}} = \frac{1}{3 + \frac{8}{17}} = \frac{1}{3 + \cfrac{1}{\frac{17}{8}}} = \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{8}}}.$$

Here we first divide both numerator and denominator of $\frac{17}{59}$ by 17. We then reduce $\frac{59}{17}$ to a mixed number, $3\frac{8}{17}$, and again divide both terms of $\frac{8}{17}$ by 8 and reduce to a mixed number, and so on.

These operations evidently produce no change in the *value* of the given fraction.

Now the several convergents of the continued fraction found are $\frac{1}{3}$, $\frac{2}{7}$, and $\frac{17}{59}$.

We find $\qquad \frac{1}{3} = \frac{19\frac{2}{3}}{59}$, too *great;*

and $\qquad \frac{2}{7} = \frac{16\frac{6}{7}}{59}$, too *small,*

but differing from the true value by only $\frac{1}{413}$.

2. If the fraction proposed had been $\frac{59}{17}$, we should have found

$$\frac{59}{17} = 3 + \frac{8}{17} = 3 + \cfrac{1}{\frac{17}{8}} = 3 + \cfrac{1}{2 + \cfrac{1}{8}};$$

and the convergents, 3, $\frac{7}{2}$, and $\frac{59}{17}$. (Art. 611.)

618. This reduction of a common to a continued fraction is evidently effected by applying to the terms of the given fraction the *process of finding the greatest common divisor,* the several *quotients* forming the successive *partial denominators.*

619. If it be required to transform any quantity whatever, as x, into a continued fraction, the nature of continued fractions will sufficiently indicate the following

RULE.—I. *Find the greatest integer contained in x, and denote it by a; and denote the fractional excess of x above a by* $\frac{1}{x_1}$. Then $x = a + \frac{1}{x_1}$. $\therefore x_1 = \frac{1}{x - a} > 1$.

II. *Find the greatest integer contained in x_1, and denote it by a_1, and denote the fractional excess of x_1 above a_1 by* $\frac{1}{x_2}$. Then

$$x_1 = a_1 + \frac{1}{x_2}.$$

III. *Apply the same process to x_2, and so on.*

Thus,

$$x = a + \frac{1}{x_1} = a + \cfrac{1}{a_1 + \cfrac{1}{x_2}} = a + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{x_3}}}, \text{ etc.}$$

If $x < 1$, we shall have $a = 0$.

We shall *always* have x_1, x_2, etc., > 1.

For if $x_1 =$ or < 1, we have $\frac{1}{x_1} =$ or > 1, and a is not the greatest integer contained in x.

620. Whenever we find a denominator, x_n, equal to a whole number, we shall have $x_n = a_n$, and the continued fraction will terminate.

This will happen if the quantity x can be exactly expressed by a common fraction.

If the quantity is not equal to a common fraction (*i. e.*, if it is incommensurable), the continued fraction will extend to infinity.

621. 1. Given $\pi = 3.14159$, employing only five decimal places. (Art. 42, 4th.) Reduce π to a continued fraction, and find approximate values.

$$\textit{Ans.}\quad \pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{25}}}}, \text{ etc.}$$

Convergents, $3, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}$, etc.

NOTE.—The second approximate value, $\frac{22}{7}$, was found by Archimedes; the fourth, $\frac{355}{113}$, by Adrian Metius.

2. The common or tropical year consists of 365.242241 mean solar days. Find approximate values for this time.

Ans. $365\frac{1}{4}$, $365\frac{7}{29}$, $365\frac{8}{33}$, $365\frac{39}{161}$, etc.

NOTE.—The third approximation shows an excess of the solar year above 365 days of $\frac{8}{33}$ of a day. To preserve the coincidence between the solar and civil year, therefore, eight years in thirty-three must contain 366 days each. That is, a day must be added to every fourth year seven times in succession, and the eighth time to the fifth year.

3. The sidereal month (*i. e.*, the time of the moon's sidereal revolution) consists of 27.321661 days, or the moon revolves 1000000 times in 27321661 days. Find approximate values of this ratio. *Ans.* $27, \frac{82}{3}, \frac{765}{28}, \frac{3907}{143}$, etc.

NOTE.—These ratios show that the moon revolves about 3 times in 82 days; 28 times in 765 days; or, more exactly, 143 times in 3907 days.

622. Continued fractions are also employed in finding the roots of equations, and in extracting the roots of numbers.

1. Extract the square root of 3; *i. e.*, find a root of the equation,

$$x^2 - 3 = 0. \qquad (1)$$

Here $$x = 1 + \frac{1}{x_1}.$$

Diminishing the roots of (1) by 1, we have,

$$y^2 + 2y - 2 = 0, \qquad (2)$$

an equation whose roots are equal to $\frac{1}{x_1}$.

Transforming (2), we find,

$$2x_1^2 - 2x_1 - 1 = 0. \tag{3}$$

This gives, $$x_1 = 1 + \frac{1}{x_2}.$$

Transforming (3) in the same manner as (1), we have,

$$x_2^2 - 2x_2 - 2 = 0, \tag{4}$$

and $$x_2 = 2 + \frac{1}{x_3}.$$

We find, in like manner,

$$2x_3^2 - 2x_3 - 1 = 0, \tag{5}$$

which being the same as (3), will have the same roots, and will give rise to transformed equations like (4) and (5).

Hence, we shall have a repetition of the equations (3) and (4), and of their roots, of which 1 and 2 are the integral parts, in endless succession.

$$\therefore \quad x = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2}}}}, \text{ etc.} = 1.732, \text{ etc.}$$

The convergents are $1,\ 2,\ \frac{5}{3},\ \frac{7}{4},\ \frac{19}{11},\ \frac{26}{15},\ \frac{71}{41},\ \frac{97}{56}$.

623. A continued fraction of this kind, in which any number of the partial denominators are *continually repeated* in the same order, is called ***Periodic.***

624. It will be found that every incommensurable root of an equation of the second degree may be expressed by a periodic continued fraction.

Of course, when the first period is found, such a fraction may be developed to any extent by simply repeating the period.

2. Extract the square root of 2.

Convergents, $1,\ \frac{3}{2},\ \frac{7}{5},\ \frac{17}{12},\ \frac{41}{29},\ \frac{99}{70}$, etc.

MISCELLANEOUS EXAMPLES AND PROBLEMS.

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1. Reduce the following fraction to its lowest terms,

$$\frac{x^3 - 5x^2 - 4x + 20}{x^3 + 5x^2 - 4x - 20}.$$

2. Add $\dfrac{3a - x}{5a + 3x}$ to $\dfrac{a + 3x}{7a + 9x}$.

3. Subtract $\dfrac{a - x}{2a^2 + 3ax + x^2}$ from $\dfrac{2a + x}{a^2 - x^2}$.

4. Reduce $\dfrac{4x + 1}{15} - \dfrac{5x - 1}{3} = x - 2$.

5. A travels 5 miles an hour, and B starts on the same road 3 hours later than A and travels $5\frac{1}{2}$ miles an hour. When will B overtake A?

6. Find the time between 5 and 6 o'clock when the hour and minute hand of a watch are together.

7. Find the square root of

$$4x^2 - 12xy + 9y^2 + 4xz - 6yz + z^2.$$

8. Find the greatest common divisor of

$$x^2 - 4, \quad x^2 + 10x + 16, \quad \text{and} \quad x^2 - 7x - 18.$$

9. Find the square root of

$$a^6 - 4a^5 + 6a^4 - 8a^3 + 9a^2 - 4a + 4.$$

10. Reduce the equation

$$(x - 3)^3 - 3(x - 2)^3 + 3(x + 1)^3 - x^3 + x - 9 = 0.$$

11. Find how much water must be mixed with 80 gallons of spirit, which cost 5 dollars a gallon, so that by selling the mixture at 4 dollars a gallon there may be a gain of 10%.

12. A person walks $7\frac{1}{2}$ miles in 2 hours $17\frac{1}{2}$ minutes, and returns in 2 hours 20 minutes. His rates of walking up-hill, down-hill, and on level road were 3, $3\frac{1}{2}$, and $3\frac{1}{4}$ miles per hour, respectively. Find the distance travelled on level road.

13. A man bought a house which cost him 4% on the purchase money to put it in repair. At the end of one year, having received no rent, he sold it for \$1192, by which he gained 10% on the original cost, besides paying him 5% on his investment as interest for the year. What did he pay for the house?

14. In a town meeting a resolution was adopted by a majority equal to $\frac{1}{3}$ of the number voting with the minority; but if 100 of those voting with the majority had voted with the minority, the majority in favor of the resolution would have been only 1. Find the number of voters on each side.

15. Reduce $\sqrt{x} - \sqrt{a} + \sqrt{x + a - b} = \sqrt{b}$.

16. Reduce $[(x - a)^2 + 2ab + b^2]^{\frac{1}{2}} = x - a + b$.

17. Reduce $\dfrac{x - \sqrt{x^2 - a^2}}{(x + \sqrt{x^2 - a^2})^{\frac{1}{2}}} = \dfrac{(x^2 + ax)^{\frac{1}{2}} - (x^2 - a^2)^{\frac{1}{2}}}{(x^2 - a^2)^{-\frac{1}{2}}}$.

18. Reduce $2x\sqrt{1 - x^4} = a(1 + x^4)$.

19. $\sqrt{x - x^{-1}} - \sqrt{1 - x^{-1}} = (x - 1)\,x^{-1}$.

20. A and B start together to walk around a circular course. In half an hour A has walked 3 complete circuits and B $4\frac{1}{2}$. Assuming that each walks at uniform speed, find when B next overtakes A.

21. The distance from A to B is 15 miles. The road is up-hill for the first 5 miles, then level for 4 miles, and then down-hill the rest of the distance. A man walks from A to B in 3 hours 52 minutes, and back in 4 hours; he then walks half way to B and back in 3 hours 55 minutes. Find his rate of walking up-hill, down-hill, and on level ground.

22. If y varies as $\dfrac{x^2}{a+x}$, and if when $x = \dfrac{b^2}{a}$, $y = \dfrac{a^3}{a^2+b^2}$, find the equations between x and y.

23. Find the sum of 9 terms of an equidifferent series whose middle term is 18.

24. Find the sum of n terms of the series,

$$\frac{1}{1+\sqrt{2}},\ \frac{1}{3+2\sqrt{2}},\ \frac{1}{7+5\sqrt{2}},\ \text{etc.}$$

25. A number consists of 3 digits. The whole number is equal to the square of the number formed by the first two digits; also the first digit exceeds twice the second by unity. What is the number?

26. Prove that the number of ways in which m positive signs and n negative signs may be placed in a row so that no two negative signs shall be together, is equal to $C_{\frac{n}{m+1}}$.

27. A and B start at the same time and travel towards each other. In 7 days A is 5 miles more than his own day's journey nearer the half-way house than B. In 10 days both have passed the half-way house and they are 100 miles apart, and B is 3 days longer than A upon the whole journey. Required their distance apart at starting and rate of walking.

28. A farmer sowed one bushel of wheat, and the second year sowed all the first year's crop, and thus continued sowing each year the whole crop of the preceding year. The 10th year the product was 1048576 bushels. What was the yearly rate of increase, on the supposition that it was the same each year?

29. Two men start from different points, at the same time, to walk towards each other; when they meet, one of them turns back, and on reaching his starting-point, again turns and walks in the same direction as at first. Each arrives at the other's starting-point at the same time. Where did they first meet? What is the ratio of their rates of walking? Where did they meet the second time?

30. A man wishes to surround a given area by hurdles. Placing them one foot apart, he lacks 80; and putting them a yard apart, he has 50 hurdles too many. How many hurdles has he, and at what distance apart must they be, so as just to enclose the space?

31. In the bottom of a cistern containing 192 gallons of water, two outlets are opened. After 3 hours, one of them is stopped, and the cistern is emptied by the other in 11 hours. Had 6 hours elapsed before the stoppage, it would have required only 6 hours more to empty it. How many gallons did each outlet discharge in an hour, supposing the discharge uniform?

32. Reduce $(4 + 5x - x^2)^{\frac{1}{2}} = 2^{\frac{1}{2}}x^{\frac{1}{2}} + (x^2 + 3x - 4)^{\frac{1}{2}}$.

33. Find the relation between the coefficients of $ax^2 + bx + c = 0$, that one root may be one-half the other.

34. Divide 111 into three parts, such that the products of the parts taken two and two may be in the ratio of 4, 5, and 6.

35. Show that the number of ways in which mn things can be divided among m persons so that each shall have n of them, is $\dfrac{\lfloor mn}{(\lfloor n)^m}$.

36. The m^{th} term of an equidifferent series is $\dfrac{1}{n}$, and the n^{th} term is $\dfrac{1}{m}$. Show that the sum of mn terms is $\dfrac{mn+1}{2}$.

37. Find the sum of n terms of the reciprocals of an equimultiple series whose first term is a and the multiplier m.

38. Find S_∞ of $2^{\frac{1}{4}}$, $4^{\frac{1}{8}}$, $8^{\frac{1}{16}}$, $16^{\frac{1}{32}}$, etc.

39. If a is an equidifferent mean between b and c, and c an harmonic mean between a and b, show that b is an equimultiple mean between a and c.

40. If n is a positive integer and x a positive fraction less than 1, show that $\dfrac{1 - x^{n+1}}{n + 1} < \dfrac{1 - x^n}{n}$.

41. If a and b are positive, and m a positive fraction less than 1, show that $(a + b)^m a^{1-m} < a + mb$.

42. Given $\left\{\begin{aligned} x^2 + y &= 7 \\ x + y^2 &= 11 \end{aligned}\right\}$ to find all the values of x and y.

43. Reduce

$$\begin{aligned} ax &= by - y^2; \\ x^2 &= y^2 + (b-y)^2. \end{aligned}$$

44. Prove that $n^5 - n$ is always divisible by 30; and, if n be odd, by 240.

45. The income of a certain railroad company would pay a dividend of 6% if there were no preferred stock; but $400000 is such stock, and is guaranteed $7\frac{1}{2}$%, the ordinary stockholders receiving only 5%. Find the amount of ordinary stock.

46. The population of a certain town in 1820 was 2375; in 1830, 2948; in 1840, 3800; in 1850, 5005; in 1860, 6636; and in 1870, 8768. By the same law of increase, find the population in 1845, 1854, 1862, and 1880.

47. A man has a plank whose ends are of unequal width. Find the distance from the narrow end that it must be cut, to make the parts equal.

48. Find the scales of the following series:

$$1, \quad 4x, \quad 18x^2, \quad 80x^3, \quad 356x^4, \quad \text{etc.}$$

$$1, \quad 2x, \quad 3x^2, \quad 8x^3, \quad 13x^4, \quad 30x^5, \quad 55x^6, \quad \text{etc.}$$

49. The population of a country increases 25% every 10 years. In what time will it double?

50. If the student who is attempting to solve this problem belongs to a class of 50, of whom $\frac{3}{10}$ cannot solve it and $\frac{4}{10}$ can solve it, and of the remainder $\frac{2}{3}$ stand 2 chances to 1 to solve it, and $\frac{1}{3}$ stand an even chance to fail, what is the chance that he will be successful?

FORMULAS.

1. $(a + x)(a - x) = a^2 - x^2.$
2. $(a + x)^2 = a^2 + 2ax + x^2.$
3. $(a - x)^2 = a^2 - 2ax + x^2.$
4. $$(a + x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{\underline{|2}} a^{n-2}x^2 + \frac{n(n-1)(n-2)}{\underline{|3}} a^{n-3}x^3 + \text{etc.}$$
5. If $x^2 + 2ax + b = 0$, $x = -a \pm \sqrt{a^2 - b}.$
6. $P_{\frac{m}{n}} = n(n-1)(n-2)\ \ldots\ (n - m + 1).$
7. $P_{\frac{n}{n}} = \underline{|n}.$
8. $$C_{\frac{m}{n}} = \frac{P_{\frac{m}{n}}}{\underline{|m}} = \frac{n(n-1)(n-2)\ldots(n-m+1)}{\underline{|m}}.$$
9. $d(ax) = adx.$
10. $d(ax - bx + c) = adx - bdx = (a - b)\,dx.$
11. $d(xyz) = xydz + xzdy + yzdx.$
12. $d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}.$
13. $d(x^n) = nx^{n-1}\,dx.$
14. $d(\log_e x) = \frac{dx}{x}.$
15. $d(\log_a x) = M_a \frac{dx}{x}.$
16. $\log_e(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \text{etc.}$
17. $\log_a(1 + x) = M_a\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \text{etc.}\right).$
18. $$\log_e(1 + z) - \log_e z = 2\left(\frac{1}{2z + 1} + \frac{1}{3(2z + 1)^3} + \frac{1}{5(2z + 1)^5} + \text{etc.}\right).$$
19. $$M_a = \frac{1}{a - 1 - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \frac{1}{4}(a-1)^4 + \text{etc.}}.$$
20. $$\log_a x = \frac{x - 1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \text{etc.}}{a - 1 - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \frac{1}{4}(a-1)^4 + \text{etc.}}.$$

21. $a = p\left(1 + \frac{nt}{1}\cdot\frac{r}{n} + \frac{nt\,(nt-1)}{1\cdot 2}\cdot\frac{r^2}{n^2} + \text{etc.}\right)\cdot\cdot$

And when $n = \infty$,

22. $a = p\left(1 + tr + \frac{t^2r^2}{\lfloor 2} + \frac{t^3r^3}{\lfloor 3} + \text{etc.}\right)$

$= pe^{tr} = p\,(2.718281)^{tr}.$

For difference series,

23. $a_n = a_1 + (n-1)\,d_1 + \frac{(n-1)\,(n-2)}{1\cdot 2}\,d_2 + \text{etc.}$

24. $S_n = na_1 + \frac{n\,(n-1)}{1\cdot 2}\,d_1 + \frac{n\,(n-1)\,(n-2)}{1\cdot 2\cdot 3}\,d_2 + \text{etc.}$

For equidifferent series,

25. $a_n = a_1 + (n-1)\,d.$

26. $S_n = \frac{a_1 + a_n}{2}\,n.$

27. $d = \frac{a_n - a_1}{n-1}.$

For equimultiple series,

28. $a_n = a_1 m^{n-1}.$

29. $S_n = \frac{a_1\,(m^n - 1)}{m-1}.$

For decreasing equimultiple series,

30. $a_\infty = a_1 m^\infty = 0.$

31. $S_\infty = \frac{a_1}{1-m}.$

32. $\frac{a}{m\,(m+p)\,(m+2p)\,\ldots\,(m+rp)} =$

$$\frac{1}{rp}\left(\frac{a}{m\,(m+p)\ldots[m+(r-1)\,p]} - \frac{a}{(m+p)(m+2p)\ldots(m+rp)}\right)$$

33. If $y^3 + py + q = 0$,

$$y = \sqrt[3]{-\frac{q}{2} + \left(\frac{q^2}{4} + \frac{p^3}{27}\right)^{\frac{1}{2}}} + \sqrt[3]{-\frac{q}{2} - \left(\frac{q^2}{4} + \frac{p^3}{27}\right)^{\frac{1}{2}}}.$$

34. $\frac{a}{0} = \infty;\quad \frac{a}{\infty} = 0;\quad \frac{0}{0} = a,$ or indeterminate.

35. $-^0 = -^2 = -^4 = -^6 = -^{2n} = +.$

36. $-^1 = -^3 = -^5 = -^7 = -^{2n+1} = -.$

NOTES.

The following brief sketches of eminent mathematicians who have made valuable contributions to our knowledge of the subjects treated in this volume and to whom reference has been made, are drawn from the most reliable sources.

NOTE I. (P. 104.)

Newton, Sir Isaac, an illustrious English philosopher and mathematician, born at Woolsthorpe, in Lincolnshire, on the 25th of December, 1642 (old style). He entered Trinity College, Cambridge, as a sub-sizar, in June, 1661, before which date it does not appear that he had been a profound student of mathematics. It has been said that he commenced the study of Euclid's Elements, but he found the first propositions so self-evident that he threw the book aside as too trifling. In 1664 he discovered the *Binomial Theorem*, in 1665 took the degree of B. A., and probably in the same year discovered the *Differential Calculus*, or *Method of Fluxions*, as he called it.

It was in the autumn of the same year that Newton conceived the idea of universal gravitation, the suggestion coming from the *fall of an apple.* It would exceed the limits of this notice even to mention his many remarkable works in Philosophy, Astronomy and Mathematics.

Near the end of his life he said, "I know not what I may appear to the world, but to myself I seem to have been only like a boy playing on the sea-shore and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me." He died at Kensington on the 20th of March, 1727, and was buried in Westminster Abbey.

Note II. (P. 186.)

M'Laurin, Colin, a Scottish mathematician, was born in Kilmodan, Argyleshire, in Feb. 1698, and died in Edinburgh, June 14th, 1746. He was a graduate of the University of Glasgow, and in 1717 was appointed Professor of Mathematics in Marischal College, Aberdeen, which position he held till 1725, when at the recommendation of Sir Isaac Newton he was called to the mathematical chair of Edinburgh. He held this professorship for over twenty years.

His principal works are, "*Geometrica Organica,*" "*A Treatise on the Percussion of Bodies,*" "*On Fluxions,*" said to be the most complete treatise on the subject and the author's most profound work; "*A Treatise on Algebra,*" and "*An Account of Sir Isaac Newton's Philosophical Discoveries.*"

Note III. (P. 191.)

Napier, John, Baron of Merchiston, was born at Merchiston Castle, near Edinburgh, Scotland, in 1550. He was educated at the University of St. Andrew's, and is celebrated as the inventor of Logarithms. His logarithmic tables were first published in 1614 under the title "*Mirifici Logarithmorum Canonis Descriptio.*" Napier also enriched the science of Trigonometry by the general theorem for the resolution of all the cases of right-angled spherical triangles. He died in 1617.

Note IV. (P. 191.)

Briggs, Henry, an eminent English mathematician, born at Warleywood, near Halifax, about 1556. He was educated at St. John's College, Cambridge. In 1596 he was chosen Professor of Geometry in Gresham College, London. He became first Savilian Professor of Geometry at Oxford in 1619.

He is chiefly distinguished for the improvement and construction of logarithms. No sooner was Napier's system of Logarithms published, than Prof. Briggs began the application of the rules in his "*Imitatio Napierea.*" He greatly improved upon

Napier's plan, by adopting 10 as the base of his system, and he has the honor of being the author of the system now in general use. He published in 1624 a work entitled "*Logarithmica Arithmetica*," containing the logarithms of all integral numbers to 20000, and also from 90000 to 100000, calculated to fourteen places. He died in 1630.

NOTE V. (P. 263.)

Sturm, Jacques Charles Francois, an eminent Swiss mathematician, was born at Geneva, in September, 1803. He was tutor to the son of Madame de Staël, with whom he visited Paris in 1823. In 1827 Sturm and his friend Colladon obtained the grand prize of Mathematics, proposed by the Academy of Sciences in Paris, for the best memoir on the compression of liquids. He discovered in 1829 the celebrated theorem which bears his name. He became Professor of Mathematics at the College Rollin in 1830, a member of the Institute in 1836, and Professor of Analysis at the Polytechnic School in 1840. He died in 1855.

NOTE VI. (P. 267.)

Horner, W. G., was an eminent English mathematician, born near the close of the last century. He was a teacher of mathematics in Bath, and died in 1837.

About fifty years ago he discovered the Method of Synthetic Division, otherwise known as the "Method of Dividing by Detached Coefficients." In 1819 he communicated to the Royal Society his method of solving algebraic equations of all degrees, entitled, "A New Method of solving Numerical Equations of all orders, by continuous Approximations." Previous to this time there was no direct and reliable method of finding the roots of equations beyond the *fourth* degree. By his method the process is comparatively brief and simple.

This method is regarded as among the most valuable contributions to the science of Mathematics in modern times. The first elementary writer that saw the value of it, says De Morgan, was Prof. J. R. Young, who

introduced it into his Treatise on Algebra, published in 1826. Prof. Young says it is the shortest method of extracting roots of higher equations that he has seen.

Note VII. (P. 282.)

Cardan, Jerome, an Italian physician and mathematician, born at Pavia in 1501. He graduated as doctor of medicine at Padua in 1525, and was successively professor of mathematics and medicine at Milan and Bologna. He dealt much in Astrology and was a professed adept in magical arts. Among his numerous writings are, "*Ars Magna,*" "*De Rerum Subtilitate,*" "*De Rerum Varietate,*" "*De Vita Propria,*" and several medical works. In 1545 he published in his "*Ars Magna*" a method of solving cubic equations, now known as "Cardan's Formula." He was the first that noticed negative roots. He died at Rome in 1576.

Note VIII. (P. 286.)

Descartes, Rene, (Lat. *Renatus Cartesius*) an illustrious French philosopher and mathematician, born at La Haye, in Touraine, March 31, 1596. He was educated at the College of La Flèche. On leaving college, at the age of nineteen, he resolved to reject all scholastic dogmas and to free himself of prejudices, and then to receive nothing that was not supported by reason and experiment. To perfect his education he determined to travel, and to this end entered the Dutch army in 1616 and came into the service of the Duke of Bavaria in 1619. In 1620 he was in the battle of Prague, but soon renounced the military profession and gave himself to more congenial pursuits. In 1637 he produced his celebrated "*Discourse on the Method of Reasoning Well and of investigating Scientific Truth,*" in which were included treatises on Metaphysics, Dioptrics and Geometry. This last treatise included the method now known as the Cartesian Geometry. The formula given in our Appendix for the reduction of biquadratic equations is due to him. He died at Stockholm in February, 1650.

ANSWERS.

Page 11, Art. 16.

1. The square of the side.
2. Twice the radius.
3. The circumference is 3.1416 times the diameter. The area is .7854 times the square of its diameter.
4. Interest is the product of principal, rate and time. Amount is the product of principal, 1 + rate and time.
5. The product.
6. The product.

Page 26, Art. 92.

1. $-$.	7. $+$.	11. Has no sign.
2. $-$.	8. $+$.	12. $+$.
3. a is $-$ and b is $+$.	9. $-$.	13. $+$.
4–6. Given.	10. $+$.	14. Has no sign.

Page 27, Art. 96.

2. $(a+b)^2 = a^2 + 2ab + b^2$.

3. $(a-b)^2 = a^2 - 2ab + b^2$.

4. $(a+b)^2 + (a-b)^2 = 2(a^2+b^2)$.

5. $a^2 - b^2 \div a - b = a + b$, or $\dfrac{a^2-b^2}{a-b} = a + b$.

6. $\dfrac{x^{\frac{1}{2}} - y^{\frac{1}{2}}}{x^{\frac{1}{4}} + y^{\frac{1}{4}}} = x^{\frac{1}{4}} - y^{\frac{1}{4}}$.

7. $(x+y)^2 - (x-y)^2 = 4xy$.

Page 28, Art. 98.

5. 10.	7. ± 2.	9. 144.
6. 4.	8. ± 2.	10. 16.

Page 31, Art. 109.

1. Given.
2. $11a + 8x - 2a^2 + ax$.
3. $a^{\frac{1}{2}} + 4a^{\frac{1}{3}} + 2ab^{\frac{1}{2}} - ab$.
4. $6a^2b + 10ab^2 - 8ax^2 - a^2x$.
5. $ab^2 - 2a^2b + a^3 - 6ac$
 $- 4ac^2 - c^3$.
6. $3xy^2 + 9x^2y + 7x^2y^2$.
7. $ax - 2bx + cx$.
8. $3a^{\frac{1}{2}}b^{\frac{1}{2}} - ab^{\frac{1}{2}} - a^{\frac{1}{2}}b$.
9. $ax^2y - 2axy^2$.
10. $\sqrt{ab} + \sqrt{ac}$.
11. $3a^2b - ab + 2b$,
 or $b\,(3a^2 - a + 2)$.
12. $y^2\,(3x + 2a - 3b)$.
13. $n\,(m - 2a + b)$.

Page 32, Art. 110.

14. Given.
15. $(a + b)^{\frac{1}{2}} - (a - b)^{\frac{1}{2}}$.
16. $(a - b - c)\sqrt{x - 1}$.
17. $\sqrt{a - x}$.

Page 33, Art. 114.

1. $4ax$.
2. $2x^2 - 3x + 7$.
3. $5ab$.
4. $2a^2 + ax$.
5. $(a - 1)(x + y)$
 $- 2by$.
6. $- 2\sqrt{x^2 + a^2}$.
7. $2b - 2c$.
8. $6x^3 + bx^2$.
9. 0.
10. 0.
11. $4ab$.
12. $2\,(b - c - m)$.
13. $- b^2$.

Page 35, Art. 119.

1. $2a - 2b - 2c - 4x + 2y - 4$.
2. $3a - 3b - 3c + 3bx - 9$.
3. $2a - 2c$.
4. $2x\,(a + b - c)$.
5. $a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}} + 2^{\frac{1}{2}}a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}}$,
 or $(2^{\frac{1}{2}} + 1)\,a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}}$.
6. $2a^2\,(c - b)$.
7. $2a^{\frac{2}{3}}b^{\frac{2}{3}} - xy^2$.
8. $2a^{\frac{1}{2}}b^{\frac{1}{2}} - 2a^{\frac{3}{2}}b^{\frac{3}{2}} - ab$.
9. $- 2c\sqrt{x + y}$.
10. $4a^{\frac{1}{2}}b^{\frac{1}{3}}$.
11. $2a + b \pm b = 2a$,
 or $2a + 2b$.
12. $4bm - 2am$.

Page 38, Art. 130.

1, 2. Given.
3. $\frac{5}{6}a^3x^2y^2z$.
4. $\frac{1}{2}a^3b^{\frac{5}{6}}cx^3$.
5. $21a^4bx^{\frac{7}{2}}y^3$.
6. $5a^{m+r+1}b^{n+s+1}xz$.
7. $6a^{\frac{m}{n}+m+2}b^{\frac{n}{m}+n+2}$
8. $3a^2bx$.
9. 4.
10. $a^{2m}b^{2n}$.
11. $a^{2m}b^{2m}$.

12. $a^{2m}b^{2m}$.

13. x^my^n.

14. $x^{2n-1}y$.

15. $x^{2n^2-2n}y^{2m^2-m}$.

16. $6a^4bx^{\frac{7}{2}}$.

17. $\mp a^4b^4x^4$.

18. $-2a^4b^2x^2$.

19. $6a^{\frac{7}{3}}x^{\frac{10}{3}}$.

20. $a^{\frac{13}{3}}x^{\frac{13}{3}}$.

Page 39, Art. 131.

21. Given.

22. $a^4 - 2a^2x^2 + x^4$.

23. $2a^3b - 2a^2bx^2 - 2ab^3$.

24. $a^5 + a^4x - ax^4 - x^5$.

25. $2a^2x + 2abx^2 + 2acx^3$
$- ax^2 - bx^3 - cx^4$.

26. $4a^2 - b^2x^2 + 2bcx^3 - c^2x^4$.

27. $a^{3m} - a^mb^{2n} + a^{2m}b^n - b^{3n}$.

28. $1 - 2x^2 + 2x^3 - 2x^5 + x^6$.

Page 40, Art. 133.

30. $3a^7b - 5a^6b^2 + 2a^5b^3 + 7a^4b^4 - 7a^3b^5 - 5a^2b^6 + 5ab^7$

31. $a^5 - 3a^3x^2 + a^2x^3 + 2ax^4 - x^5$.

32. $x^6 + x^4y^2 - x^2y^4 - y^6$.

33. $a^8 - a^7x - 5a^4x^4 + 8a^3x^5 - 4a^2x^6 + ax^7$.

Page 44, Art. 144.

5. $4ab^{-3}c^{-1}d^2$, or $\dfrac{4ad^2}{b^3c}$.

6. $\frac{2}{3}a^{-1}b^{-1}cd^{-2}$, or $\dfrac{2c}{3abd^2}$.

7. $\dfrac{3a^2c^5d^2}{b^3}$.

8. $\dfrac{3a^5y^2}{b^2x^6}$.

Page 45, Art. 145.

9. Given.

10. $-4a^{m-1}b^{n-1}$.

11. $-2a^4b^2$.

12. $5a^mb^{-n}$, or $\dfrac{5a^m}{b^n}$.

13. $-4ab^2$.

14. $-\frac{1}{4}a^{-1}b^{-2}$, or $-\dfrac{1}{4ab^2}$.

15. $-3a^{\frac{1}{4}}b^{\frac{1}{6}}$.

16. $ab^{\frac{1}{4}}$.

17. $-\dfrac{a}{b} = -ab^{-1}$.

18. $-a^{m^2}b^{n^2}$

19. $-a^{m^2-m}b^{n^2-n}$.

20. $x^{n+1}y^{m-1}$.

21. $a^{n+2}b^{n+1}x^n$.

Page 46, Art. 146.

1. $bx - 2ax^2 + 3a^2bx^3$.

2. $2 + a^nb^{2m} - 3a^{3n}b^{2m}$.

3. $3a^{m-n} - 2 + 4a^{2m-n}b^{2n-m}$.

4. $2a^mb^{s-2n} - 3a^{m+2n}b^s$.

5. $2a^{m-2n}b^{s} - 3a^{m}b^{s+2n}$.

6. $2a^{2(m-n)}b^{2(s-n)} - 3a^{2m}b^{2n}$, and $2a^{-n}b^{-n} - 3a^{n}b^{n}$.

7. $\dfrac{x^{y-1}}{y} - \dfrac{y^{x-1}}{x} = x^{y-1}y^{-1} - x^{-1}y^{x-1}$.

8. $\dfrac{a^{\frac{n}{m}}}{b} + \dfrac{b^{\frac{m}{n}}}{a}$.

9. $\dfrac{a^{\frac{mn-m}{r}}}{b^{\frac{m}{s}}} + \dfrac{b^{\frac{mn-m}{s}}}{a^{\frac{m}{r}}}$.

Page 48, Art. 148.

1, 2. Given.

3. $x^4 - 8x^2 + 4x - 1$.

4. $x^6 + x^4y^2 + x^2y^4 + y^6$.

5. $x^4 - a^2x^2 + a^4$.

6. $a^{\frac{3}{4}} + a^{\frac{1}{2}}x^{\frac{1}{4}} + a^{\frac{1}{4}}x^{\frac{1}{2}} + x^{\frac{3}{4}}$.

7. $a + a^{\frac{1}{2}}x^{\frac{1}{2}} - x$.

8. $x^n + a^n$.

9. $a^nb^n - a^{n-1}b^{2n}$.

10. $a^{-\frac{2}{3}} - a^{-\frac{1}{3}}b^{-\frac{1}{3}} + b^{-\frac{2}{3}}$

11. $a^3 - 5a^2x - ax$.

12. x.

13. $1 - 2x + x^2$.

14. $b^2 + 2bx + x^2$.

15. $x^3 + 3x^2y + 3xy^2 + 9y^3 + \dfrac{32y^4}{x - 3y}$.

16. $2a^2 - 2a - 2 + \dfrac{2a - 13}{3a^2 - 2a + 1}$

17. $2a^2 + 5ax + x^2$.

18. Given.

Page 50, Art. 150.

1. Given.

2. $a^2 - 2ax + x^2$.

3. $a - b$.

4. $a^2 - ax + x^2$.

5. $x^3 - 3x^2 + 3x - 1$.

6. $x^3 - 2x^2y + 2y^3$.

7. $1 - 2b + 3b^2 - 4b^3$.

8. $2a^2 + 5ax + x^2$.

Page 51, Art. 151.

5. Given.

6. $a^4 + a^3b + ab^3 + b^4$.

7. $x^2 + 2xy + y^2$.

8. $a^3 + 3a^2x + 3ax^2 + x^3$.

Page 52, Art. 151.

12. $x^5 + x^4 - 2x^3 - 2x^2$.

13. $x^4 + 10x^3 + 30x^2 + 87x + 268 + \dfrac{799}{x - 3}$.

14. $x^3 - x^2 - 2x + 4 - \dfrac{8}{x - 2}$. Also $x^3 + 8$.

15. $x^4 - x^3 - 4x^2 + 6x - 6 - \dfrac{1}{x + 1}$.

16. $x^5 - 3x^4 + 4x^3 - 12x^2 + 36x - 106 + \dfrac{317}{x + 3}$.

17. $x^5 - 3x^4 + 3x^3 + 3x^2 - 3x + 1$;
Also $x^5 - 5x^4 + 11x^3 - 11x^2 + 5x - 1$.

18. $x^4 - 2x^3 - 3x^2 + 6x - 6 + \dfrac{7}{x + 2}$;

Also $x^4 + 3x^3 + 2x^2 + 6x + 24 + \dfrac{67}{x - 3}$.

19. $x^2 + x + 1$; Also $x^2 - 2x + 4 - \dfrac{9}{x + 2}$.

20. $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$;

Also $x^6 - x^5 + x^4 - x^3 + x^2 - x + 1 - \dfrac{2}{x + 1}$.

21. $a^8 - a^4x^4 + x^8$; Also $a^9 - a^6x^3 + a^3x^6 - x^9 + \dfrac{2x^{12}}{a^3 + x^3}$.

22. $x^8 - x^6y^2 + x^4y^4 - x^2y^6 + y^8$; Also $x^5 - y^5 + \dfrac{2y^{10}}{x^5 + y^5}$.

23. $x^{12} - x^{10} + x^8 - x^6 + x^4 - x^2 + 1$;

And $x^7 - 1 + \dfrac{2}{x^7 + 1}$.

24. $x^{12} - a^{12}x^6 + a^{24}$; And
$x^{16} - a^4x^{14} + a^8x^{12} - a^{12}x^{10} + a^{16}x^8 - a^{20}x^6 + a^{24}x^4 - a^{28}x^2 + a^{32}$.

25. $x^6 - x^3y^4 + y^8$; And $a^4 - a^3b^2 + a^2b^4 - ab^6 + b^8$.

26. $x^4 - 3x^3 - 4x^2 + 19x - 61 - \dfrac{174}{x - 3}$;

And $x^4 - 9x^3 + 32x^2 - 103x + 305 + \dfrac{924}{x + 3}$.

27. $x^6 - x^5 + 4x^4 - 6x^3 + x^2 - x + 4 - \dfrac{3}{x + 1}$;

And $x^6 + x^5 + 4x^4 + 2x^3 - 3x^2 - 3x + \dfrac{1}{x - 1}$.

28. $x^5 + 3x^4 - 10x^3 + 12x^2 - 19x + 23 - \dfrac{7}{x + 1}$;

And $x^5 + 5x^4 - 2x^3 - 7x - 3 + \dfrac{13}{x - 1}$.

Page 55, Arts. 155-159.

To give the answers to problems in *Factoring* would destroy their value to the student. They are therefore omitted. The same reason may be inferred when other answers are omitted.

Page 61, Art. 168.

1, 2. Given.
3. $a(x+a)^2$.
4. $x+5$.
5. $b(a+b)$.
6. $x(a+b)$.
7. $a-b$.
8. $x-1$.
9. $a-x$.
10. $(x-1)^2(x-2)$.
11. $x+2$.
12. $3(x+1)$.
13. $x+3$.
14. $x+2$.
15. x^2-4.

Page 62, Art. 173.

1. $x^4+ax^3-a^3x-a^4$.
2. $6a^3x^4y^5$.
3. $a^3+a^2x-ax^2-x^3$.
4. x^3+x^2-x-1.
5. x^4-x^3-x+1.
6. $x^{m+n}y^{m+n}$.
7. x^4-2x^2+1.
8. $a^4-a^3x+ax^3-x^4$.
9. $a^4+a^3x-ax^3-x^4$.
10. $a^4-2a^2x^2+x^4$.
11. a^4-1.
12. $x^{10}-x^9y+2x^8y^2-x^7y^3+x^6y^4-x^4y^6+x^3y^7-2x^2y^8+xy^9-y^{10}$.
13. $x^8-a^2x^6+a^6x^2-a^8$.
14. $x^4+2x^3-9x^2-2x+8$.
15. $x^4+5x^3+5x^2-5x-6$.

Page 66, Art. 193.

1. Given.
2. $\frac{b^2c}{6a^2d^2e}$.
3. $\frac{2}{3abc}$.
4. $\frac{x^2-xy+y^2}{x-y}$.
5. $\frac{x^4-x^2+1}{x^2-1}$.
6. $\frac{1}{x^4-x^2y^2+y^4}$.
7. $\frac{x-7}{x+3}$.
8. $\frac{2}{3(x^2+1)}$.
9. $\frac{x-a}{x+a}$.

Page 66, Art. 194.

1. $2yz$.
2. $b^2+\frac{b^2(b-1)}{1-a^2}$.
3. a^2+ab+b^2.
4. $a^8-a^6b^2+a^4b^4-a^2b^6+b^8$.
5. $a^{12}-a^6b^6+b^{12}$.
6. ax^2+x.
7. $x+1-\frac{x+1}{x^2+x-12}$.
8. $b^2+\frac{b-1}{a+1}$.

Page 67, Art. 195.

2. $-\dfrac{b^3}{a}$.

3. $\dfrac{x}{ab}$.

4. $\dfrac{2x^2+2}{x+1}$.

5. $-\dfrac{4ax}{a-x}$.

6. $\dfrac{2ab}{a+b}$.

7. 0.

Page 68, Art. 196.

1. $\dfrac{a^4+a^2-2}{a^2c-c}$.

2. $\dfrac{a^6-2a^4+2a^2-1}{a^6+1}$.

3. $\dfrac{a^4+2a^3+2a^2+2a+1}{a^4-1}$.

4. $\dfrac{x^6+5x^5+5x+25}{x^2-2x-35}$.

5. $\dfrac{\dfrac{a^3+a^2+a}{a+1}}{a^3-1}$.

6. $\dfrac{x^3-1}{x^2-1}$.

Page 69, Art. 198.

1. $\dfrac{x^4-x^3y+x^2y^2-xy^3}{x^4-y^4}$; $\dfrac{x^4+x^2y^2}{x^4-y^4}$; $\dfrac{x^3}{x^4-y^4}$.

2. $\dfrac{x^4+1}{x^8-1}$; $\dfrac{2x^4-2}{x^8-1}$; $\dfrac{3x^6-3x^4+3x^2-3}{x^8-1}$.

3. $\dfrac{4(a^7+a^6-a-1)}{8(a^8+a^6-a^2-1)}$;

$\dfrac{2(a^7-a^6+2a^5-2a^4+2a^3-2a^2+a-1)}{8(a^8+a^6-a^2-1)}$;

$\dfrac{a^4-1}{8(a^8+a^6-a^2-1)}$.

4. $\dfrac{4(x^2-4)}{x^4-5x^2+4}$; $\dfrac{x^3-3x^2+4}{x^4-5x^2+4}$; $\dfrac{x^3+3x^2-4}{x^4-5x^2+4}$.

5. C. D. is $(x^4-1)(x^3+3x^2-3x+3)(x^2+3)$.
Numerators are
$x(x^3+3x^2-3x+3)(x^2+3)$;
$(x^2+1)^2(x-1)(x^2+3)$;
And $(x^4-1)(x^3+3x^2-3x+3)$.

6. $\dfrac{x+y}{x^4-y^6}$; $\dfrac{(x-y)(x^2-y^3)}{x^4-y^6}$; $\dfrac{(x^2+y^2)(x^2+y^3)}{x^4-y^6}$.

Page 70, Art. 199.

1, 2. Given.

3. $\frac{3a(a+x)}{b^2}$.

4. $\frac{2(a+b)}{a-b}$.

5. 0.

6. -1.

7. $\frac{x+4}{x-4}$.

8. $\frac{a^2}{x+a^2}$.

9. $\frac{-3}{x^3+2x^2-9x-18}$.

10. $\frac{-x^2}{x^4+x^3-x-1}$.

Page 71, Art. 201.

1. $\frac{x}{a+x}$.

2. $\frac{1}{x+1}$.

3. $\frac{1}{1-x}$.

4. 5.

5. $\frac{x+4}{x-3}$.

6. $\frac{x}{x+4}$.

7. $\frac{1}{x^2-a^5}$.

8. $\frac{10}{x^3-ax^4+a^2}$.

Page 72, Art. 202.

1. $\frac{1}{a-x}$.

2. $\frac{x}{a+b^2}$.

3. $\frac{3}{x^3-1}$.

4. $\frac{1}{(a^2-b^3)^2}$.

5. $\frac{x^6-a^2x^3+a^4}{x^4-a^3x^2+a^6}$.

6. $\frac{x^5+a^3}{x^3+a^5}$.

7. $\frac{ax+7x+4a+28}{ax+5x+6a+30}$.

8. $\frac{a-1}{(x^2-x+1)(a^2-a+1)}$.

9. $-(x^5+x^4+x^3+x^2+x+1)$.

Page 72, Art. 203.

1. $\frac{b}{3xyz^2}$.

2. $\frac{2xyz^2}{9ac^2d}$.

3. $\frac{x^2-3x+9}{x^2-4}$.

4. $\frac{x}{c}$.

5. $\frac{4+a}{(1-a^2)(3-a^2)}$.

6. $\frac{a^2+ay+y^2}{x^2-y^2}$.

Page 73, Arts. 205, 206.

1. $\dfrac{25a^2 - 5ab}{b} = \dfrac{25a^2}{b} - 5a.$ 2. $3a(1 - x).$

1. $\dfrac{acex}{3by}.$

2. $\dfrac{c^{10} + c^5x^6 + x^{12}}{a^4 - a^2x^2 + x^4}.$

3. $\dfrac{b^2(c^3 + d^3)}{a(ab^2 + c^2d)}.$

4. $\dfrac{(y^8 - y^4 + 1)(a + 1)}{a^4 + a^2 + 1}.$

5. $\dfrac{a^4(x^{12}y^{12} - x^{10}y^{10} + x^8y^8 - x^6y^8 + x^4y^4 - x^2y^2 + 1)}{y^6(y^6 - ay^5 + a^2y^4 - a^3y^3 + a^4y^2 - a^5 + a^6}.$

6. $\dfrac{c^2 - 5c - 14}{c^2 - 169}.$

7. $\dfrac{a^2 - 4}{a^2 - 8a + 16}.$

8. $\dfrac{a^6 + a^3y^7 + y^{14}}{a^4 - y^5}.$

Page 75, Art. 210.

1. $\dfrac{a^3 - a^2b + 3ab^2 + b^3}{a(a^2 - b^2)}.$

2. $\dfrac{2a^2 - 2a + 1}{a(a - 1)}.$

3. $-\dfrac{ax}{a + x}.$

4. $\dfrac{a^4 + 6a^2 + 1}{a^4 - 1}.$

5. $\dfrac{1}{a - 1}.$

6. $\dfrac{x^2 + ax + a^2}{x - a}.$

7. $\dfrac{2ax}{a + x}.$

8. $\dfrac{a - x}{ab}.$

9. $\dfrac{1}{x - 1}.$

10. $a^2 - x^2.$

11. $a + x.$

12. $\dfrac{a^2b^2 - a^2 - b^2 + 1}{a^2 - b^2}.$

13. $1.$

14. $x - \dfrac{1}{x}.$

15. $a.$

16. $1 + \dfrac{1}{x^2}.$

17. $a - 1.$

18. $-\dfrac{1}{x + y}.$

Page 82, Art. 233.

1–5. Given.

6. $x = 12.$

7. $x = 7.$

8. $x = \dfrac{abc}{ab + ac - bc}.$

9. $x = \dfrac{b - c}{a - b}.$

10. $x = \dfrac{mn - ab}{a + b - m - n}$.

11. $x = \dfrac{a^2c + ab^2 + bc^2 - (a + b + c)}{ab + ac + bc - 1}$.

12. $x = 5$.

13. $x = \dfrac{bn - am}{m - n}$.

14. $x = 5$.

15. $x = 7$.

16. $x = 4$.

17. $x = 6$.

18. $x = 4$.

19. $x = 7b$.

20. $x = \frac{1}{2}b\,(1 - a^2)$.

21. $x = \dfrac{c + 2(a - b)}{2b}$.

22. $x = 2$.

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PROBLEMS.

1. 2 dols., 6 halves, 30 qrs., 90 dimes, and 450 half dimes.
2. 36 years.
3. 210 acres.
4. $\dfrac{s + d}{2}$ = the greater, and $\dfrac{s - d}{2}$ = the less.
5. $\dfrac{a\,(m + 1)}{2}$ = one part, and $\dfrac{a\,(1 - m)}{2}$ = the other part.
6. The parts are $\dfrac{an - cn + b}{n + 1}$ and $\dfrac{cn + a - b}{n + 1}$.
7. \$80000.
8. — \$600.
9. 1st, 6 miles; 2d, 15 miles; 3d, ∞ miles.

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10. a years.

11. \$93.

12. $\dfrac{5280mc}{5280m - nc}$; c being feet.

13. $\dfrac{s}{1 + a}$.

14. $\dfrac{abc}{ab + ac + bc}$ days.

15. 40 gallons.

16. $\dfrac{c - b}{a}$.

17. $\dfrac{a}{n - m}$.

18. $\dfrac{a}{m + n}$.

19. Brandy 30, wine 40, and water 70 gallons.

20. A's, \$83$\frac{1}{3}$; B's, \$91$\frac{2}{3}$; C's, \$83$\frac{1}{3}$.

21. 30 and 7. Prob. 4 gives the formula.

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22. 60 apples and 20 oranges.

23. $x = \frac{a - bn}{m - n}$; in which x = original number of oranges, mx = the number of apples, a = number of apples, and b = number of oranges sold, and n = ratio of oranges to apples after the sale.

24. $\frac{2a(1 - mn)}{mn + m - 2}$ = No. sold in all.

$\frac{am(1 - n)}{mn + m - 2}$ = No. of apples at first.

$\frac{a(1 - n)}{mn + m - 2}$ = No. of oranges at first.

25. 1st, 140 qts.; 2d, 60 qts.; 3d, 45 qts.; 4th, 80 qts.
26. 70; 25; 36; 15; 20.
27. $1\frac{7}{9}$ of a mile per hour.
28. 1 mile per hour.
29. 1000.

Page 95, Art. 251.

1. Given.
2. $x = 2, \quad y = 3.$
3. $x = 12, \quad y = 18.$
4. $x = \left(\frac{bm + dn}{ad + bc}\right) ac, \quad y = \left(\frac{am - cn}{ad + bc}\right) bd.$
5. $x = 11, \quad y = 9.$
6. $x = \frac{h - bd}{a - b}, \quad y = \frac{ad - h}{a - b}.$
7. $x = h^2(a + b), \quad y = h(a + b).$
8. $x = 7\frac{1}{2}, \quad y = 5.$
9. $x = 244, \quad y = -172.$
10. $x = 8, \quad y = 12.$
11. $x = \frac{4a + b}{6}, \quad y = \frac{b - 2a}{12}.$
12. $x = \frac{b'c - bc'}{ab' - a'b}, \quad y = \frac{a'c - ac'}{a'b - ab'}.$
13. The equations are not independent and represent but one condition; viz., that $3x = 4y$.

14. $x = \dfrac{b' - b}{a - a'}$, $y = \dfrac{ab' - a'b}{a - a'}$.

15. $x = \infty$, $y = \infty$. The two conditions are incompatible except for infinite quantities.

16. $x = 3$, $y = 4$.

17. $x = \dfrac{c + d}{2}$, $y = \dfrac{c - d}{6}$.

18. $x = 6$, $y = 4$.

19. $x = 4$, $y = 15$.

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PROBLEMS.

1. The number is 462.

2. 6 oranges and 10 apples.

3. The fraction is $\frac{12}{19}$.

4. The 1st in 12, the 2d in 20, and the 3d in 30 hours.

5. The problem is indeterminate, only one condition being given from which to determine two unknown quantities, viz., that the interest on the sum for 2 months shall be $10.

6. A in 25 days and B in $16\frac{2}{3}$ days.

7. A, $\frac{2}{5}$; B, $\frac{4}{15}$; and C, $\frac{1}{3}$. C worked 6 days.

8. $\dfrac{ac - ab}{a - b}$ of the 1st, and $\dfrac{ab - bc}{a - b}$ of the 2d.

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9. Loaves 6 cts. each; Fruit 20 cts.

10. The fraction is $\frac{4}{5}$.

11. 1st, $\dfrac{(m^2 + mn)\,c - (a^2 + ab)\,p}{(a + b)(m + n)(mb - an)}$;

2d, $\dfrac{(n^2 + mn)\,c - (b^2 + ab)\,p}{(a + b)(m + n)(an - mb)}$.

12. The problem is indeterminate.

13. $400 and $100, at 2% and 4%.

14. The fraction is $\frac{3}{20}$.

15. A's, 500 dollars; B's, — 500 dollars.

16. A, $900; B, $600. The mortgage $500.

NOTE.—The student will observe that the preceding problem contains really two independent problems.

Page 99, Art. 252.

1. Given.
2. $x = 3, \quad y = 2,$ and $z = 1.$
3. $x = \dfrac{(a^2 - bc)\,m + (b^2 - ac)\,n + (c^2 - ab)\,r}{a^3 + b^3 + c^3 - 3abc}$

   $y = \dfrac{(b^2 - ac)\,m + (c^2 - ab)\,n + (a^2 - bc)\,r}{a^3 + b^3 + c^3 - 3abc},$

   $z = \dfrac{(c^2 - ab)\,m + (a^2 - bc)\,n + (b^2 - ac)\,r}{a^3 + b^3 + c^3 - 3abc}.$
4. $x = 5, \quad y = 7, \quad z = 4, \quad u = 3.$
5. $x = 2, \quad y = 3, \quad z = 4.$
6. $x = 1, \quad y = 1, \quad z = 2, \quad u = 2.$
7. $x = \dfrac{2abc}{ab + bc - ac}, \quad y = \dfrac{2abc}{ac + bc - ab},$

   $z = \dfrac{2abc}{ab + ac - bc}.$
8. $x = 0, \quad y = 0, \quad z = 0.$
9. $x = \dfrac{(a + b)\,m + cn}{(a + b)^2 - c^2}, \quad y = \dfrac{(a + b)\,n + cm}{(a + b)^2 - c^2}.$
10. $x = \dfrac{2ac}{a + c}, \quad y = \dfrac{2bc}{b + c}, \quad z = \dfrac{2ab}{a + b}.$

PROBLEMS.

1. 1257.
2. 40, 60, 70.
3. A's age $= 3a - 2c$, B's $= 2b + 2c - 3a$,

   C's $= 3a - 2b$.

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4. 40, 60, 50.
5. $\$122\frac{11}{12}$ and $\$97\frac{11}{12}$, value of Horses;

   $\$32\frac{1}{12}$ and $\$12\frac{1}{12}$, value of Saddles.
6. A, \$800; B, \$900; C, \$600.

7. $$\text{A's} = a\frac{m'm''n''(n'-1)(n-mn')-mm'm''(n-n')(n'n''-1)}{n''(m''n'-m')(n-mn')-mn''(m'n-n')(n'n''-1)},$$

$$\text{B's} = a\frac{mn''(n-n)(m''n'-m')-mm''n''(n'-1)(m'n-n')}{n''(n-mn')(m''n'-m')-mm''(n'n''-1)(m'n-n')},$$

$$\text{C's} = a\frac{mn'n'(m'n''-m'')(1-m')-nn'n''(m'-m'')(1-mm')}{m(m'n-m'')(n'-m'n)-nn''(m'-m''n')(1-mm')}.$$

8. A, $\dfrac{2abd}{ab+bd-ad}$; B, $\dfrac{2abd}{ad+bd-ab}$;

C, $\dfrac{2abd}{ad+ab-bd}$; D, $\dfrac{2abcd}{abd-acd+bcd}$.

9. A's, \$1000; B's, — \$500; C's, \$0.

10. 3, 5, and 6 miles.

Page 106, Art. 274.

5. $a^2+2ax+x^2$ and $a^2-2ax+x^2$.

6. $4a^2+4ab+b^2$ and $a^2-4ab+4b^2$.

7. $a^{\frac{1}{2}}+\dfrac{b}{2a^{\frac{1}{2}}}-\dfrac{b^2}{8a^{\frac{3}{2}}}+$ etc. and $a^{\frac{1}{2}}-\dfrac{b}{2a^{\frac{1}{2}}}-\dfrac{b^2}{8a^{\frac{3}{2}}}-$ etc.

8. $x^6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6$;
$x^7-7x^6y+21x^5y^2-35x^4y^3+35x^3y^4-21x^2y^5+7xy^6-y^7$.

9. $32a^5+160a^4b+320a^3b^2+320a^2b^3+160ab^4+32b^5$;
$27a^3-81a^2b+81ab^2-27b^3$.

10. $\dfrac{1}{a}-\dfrac{x}{a^2}+\dfrac{x^2}{a^3}-\dfrac{x^3}{a^4}+$ etc.

$\dfrac{1}{a^2}+\dfrac{2x}{a^3}+\dfrac{3x^2}{a^4}+\dfrac{4x^3}{a^5}+$ etc.

11. $a^{\frac{2}{3}}+\dfrac{2x}{3a^{\frac{1}{3}}}-\dfrac{x^2}{9a^{\frac{4}{3}}}+\dfrac{4x^3}{81a^{\frac{7}{3}}}-$ etc.;

$a^{\frac{2}{3}}-\dfrac{2x}{3a^{\frac{1}{3}}}-\dfrac{x^2}{9a^{\frac{4}{3}}}-\dfrac{4x^3}{81a^{\frac{7}{3}}}-$ etc.

12. $a^{\frac{5}{2}}+\frac{5}{2}a^{\frac{3}{2}}c+\frac{15}{8}a^{\frac{1}{2}}c^2+\dfrac{5c^3}{16a^{\frac{1}{2}}}-\dfrac{5c^4}{64a^{\frac{3}{2}}}+$ etc.;

$\dfrac{1}{a^{\frac{5}{2}}}-\dfrac{5c}{2a^{\frac{7}{2}}}+\dfrac{35c^2}{8a^{\frac{9}{2}}}-\dfrac{105c^3}{16a^{\frac{11}{2}}}+$ etc.

13. $\frac{1}{a^{\frac{5}{2}}} + \frac{5c}{2a^{\frac{7}{2}}} + \frac{35c^2}{8a^{\frac{9}{2}}} + \frac{105c^3}{16a^{\frac{11}{2}}} + \text{etc.};$

$a^{\frac{5}{2}} - \frac{5}{2}a^{\frac{3}{2}}c + \frac{15}{8}a^{\frac{1}{2}}c^2 - \frac{5c^3}{16a^{\frac{1}{2}}} - \frac{5c^4}{128a^{\frac{3}{2}}} - \text{etc.}$

14. $a^{\frac{1}{2}}x^{\frac{1}{2}} + \frac{by}{2a^{\frac{1}{2}}x^{\frac{1}{2}}} - \frac{b^2y^2}{8a^{\frac{3}{2}}x^{\frac{3}{2}}} + \text{etc.};$

$\frac{1}{a^{\frac{1}{2}}x^{\frac{1}{2}}} + \frac{by}{2a^{\frac{3}{2}}x^{\frac{3}{2}}} + \frac{3b^2y^2}{8a^{\frac{5}{2}}x^{\frac{5}{2}}} + \text{etc.}$

15. $a^2 + 4ab - 8ac + 4b^2 - 16bc + 16c^2.$

16. $4a^2 - 12a^2x + 4ab^2 - 4axy + 4az + 9a^2x^2 - 6ab^2x + 6ax^2y - 6axz + b^4 - 2b^2xy + 2b^2z + x^2y^2 - 2xyz + z^2.$

17. $a^2x^2 + 2abxy - 6axz + 10ax + b y^2 - 6byz + 10by + 9z^2 - 30z + 25.$

18. $a^3 + 3a^2b - 3a^2c + 3a^2d - 6bcd + b^3 + 3ab^2 - 3b^2c + 3b^2d - 6acd - c^3 + 3ac^2 + 3bc^2 + 3c^2d + 6abd + d^3 + 3ad^2 + 3bd^2 - 3cd^2 - 6abc.$

19. $8x^3 - 36x^2y + 12x^2z - 27y^3 + 54xy^2 + 27y^2z + z^3 + 6xz^2 - 9yz^2 - 36xyz.$

20. $a^3x^3 + 3a^2bx^2y + 3a^2x^2z + 3a^2mx^2 - 3a^2nx^2 + 6bmyz - 6mnz - 6bmny - 6bnyz + b^3y^3 + 3ab^2xy^2 + 3b^2y^2z + 3b^2my^2 - 3b^2ny^2 + 6amxz - 6amnx - 6anxz + z^3 + 3axz^2 + 3byz^2 + 3mz^2 - 3nz^2 + 6abmxy - 6abnxy + m^3 + 3am^2x + 3bm^2y + 3m^2z - 3m^2n + 6abxyz - n^3 + 3an^2x + 3bn^2y + 3n^2z + 3mn^2.$

Page 109, Art. 278.

1. $a^{\frac{1}{3}} + x^{\frac{1}{3}}.$
2. $a^{\frac{1}{3}} - x^{\frac{1}{3}}.$
3. $a^n \pm x^n.$
4. $a^2 + 2ax + x^2.$
5. $a^3 + 3a^2x + 3ax^2 + x^3 =$ sq. root; $a^2 + 2ax + x^2 =$ cube root.
6. $a^2 + x^2.$
7. $a^{\frac{1}{2}} + \frac{1}{2}.$
8. $a - x.$
9. $a - 1.$

11. $3 - \sqrt{2}.$
12. $6 \pm \sqrt{5}.$
13. $5 \pm 2\sqrt{2}.$

Pages 119-122, Arts. 305-310.

2. $3\left(\frac{1}{2} + \frac{1}{2}\sqrt{-3}\right)$.
3. $2\left(\frac{1}{2} \pm \frac{1}{2}\sqrt{-3}\right)$.
4. -5 and $5\left(\frac{1}{2} \pm \frac{1}{2}\sqrt{-3}\right)$.
6. a.
7. $5\sqrt{ax} - 2\sqrt{-ax}$.
8. $2\sqrt{a^2 - x^2}$.

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9. $2x$.
10. $a^3b + ab^3 - a^2b + ab^2 + a^{\frac{5}{2}}b^{\frac{3}{2}} + a^{\frac{3}{2}}b^{\frac{5}{2}}$.
11. $a^{\frac{1}{3}} + b^{\frac{1}{3}}$.
12. $a^{\frac{1}{2}} + b^{\frac{1}{2}}$.
13. $a - b$.
14. $a - b$.
15. $(a+b)(a^{\frac{1}{2}}+b^{\frac{1}{2}})(a^{\frac{1}{4}}+b^{\frac{1}{4}})$.
16. $2a^{\frac{1}{2}}b^{\frac{1}{2}} - 5ab^{\frac{1}{2}}$
17. $\sqrt{-abcd}$.
18. $8a^3$.
19. $a^2 + x$.
20. $a + \sqrt{-x}$.
21. $\sqrt{-x} - \sqrt{-a}$.
22. $a\left(a^{\frac{1}{2}} - x^{\frac{1}{2}}\right)$.
23. $a^{\frac{1}{2}} + x^{\frac{1}{2}}$.
24. $a^{\frac{m^2+n^2}{mn}}b^{\frac{m^2+n^2}{mn}} + a^{\frac{1+n}{m}}b^{\frac{1+m}{n}} + a^{\frac{1+m}{n}}b^{\frac{1+n}{m}} + a^{\frac{m+n}{mn}}b^{\frac{m+n}{mn}}$.
25. $a^n - b^n$.
26. $a^2 + ab^{\frac{1}{3}} + b^{\frac{2}{3}}$.
27. $a^{\frac{4}{3}} - a^{\frac{2}{3}}b + b^2$.
28. $a^{\frac{5}{3}} + a^2b^{\frac{1}{3}} + a^{\frac{3}{2}}b^{\frac{2}{3}} + ab + a^{\frac{1}{3}}b^{\frac{4}{3}} + b^{\frac{5}{3}}$.
29. $a^{\frac{16}{5}} - a^{\frac{12}{5}}b^{\frac{2}{5}} + a^{\frac{8}{5}}b^{\frac{4}{5}} - a^{\frac{4}{5}}b^{\frac{6}{5}} + b^{\frac{8}{5}}$.
30. $a^{\frac{22}{3}} - a^{\frac{20}{3}}b^{\frac{1}{4}} + a^6b^{\frac{1}{2}} - a^{\frac{16}{3}}b^{\frac{3}{4}} + a^{\frac{14}{3}}b - a^4b^{\frac{5}{4}} + a^{\frac{10}{3}}b^{\frac{3}{2}} - a^{\frac{8}{3}}b^{\frac{7}{4}} + a^2b^2 - a^{\frac{4}{3}}b^{\frac{9}{4}} + a^{\frac{2}{3}}b^{\frac{5}{2}} - b^{\frac{11}{4}}$.
31. 1024.
32. 1024.
33. 1024.
34. 1024.

Page 125, Art. 312.

2. $2\sqrt{3}$.
3. 11.
4. -12.
5. $-6\sqrt{2}$.

Art. 313.

3. $3\sqrt{2}$.
4. $\frac{2}{3}\sqrt{3}$.
5. $\sqrt{2}$.
6. $1 + \sqrt{-2}$.

Page 128, Art. 318.

1. a.
2. 1.
3. $\dfrac{2\sqrt{1+x}}{x} - 2$.
4. $\dfrac{4ax^2}{a^2-x^2}$.
5. 13.
6. $2 + 3\sqrt{3} - \sqrt{5}$.

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7. $-\sqrt{2} + \frac{1}{2}\sqrt{-1}$.
8. Zero.
9. $\dfrac{2a^2}{b^2}$.
10. $\dfrac{2a}{x}$.
11. Zero.
12. $-\frac{1}{3}\sqrt{-30}$.
13. -2.
14. $-\dfrac{1}{8x\sqrt{1-x}}$.
15. Zero.
16. $\dfrac{\sqrt{(1+x^2)^{\frac{1}{2}} + x}}{2(1+x^2)^{\frac{1}{2}}} = \frac{1}{2}\sqrt{1 + \dfrac{x}{(1+x^2)^{\frac{1}{2}}}}$.
17. $(ax + a^2x^2 + a - x)\sqrt{a-x}$.

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18. $\dfrac{ax}{x^2-a^2}$.
19. $a - x$.
20. $a + 2x - b$.
21. $abc^2\sqrt{-a}$.
22. 1.
23. $b^{\frac{1}{12}}c^{\frac{3}{2}}$.
24. $a^{m+n}b^{m+n}c^{s+2}$.
25. $a^{\frac{(m+n)^2}{mn}}b^{\frac{(m+n)^2}{mn}}c^{\frac{m(n^2+1)}{n}}$
26. $a^2 + \dfrac{a^4}{b}$.
27. $\dfrac{a(a^{\frac{1}{2}} + b^{\frac{1}{2}})}{a-b}$.
28. $\dfrac{x^2 - xy^{\frac{1}{3}} + y^{\frac{2}{3}}}{x^3+y}$.
29. $\dfrac{b^3(a^{\frac{5}{3}} + a^{\frac{4}{3}}b^{\frac{1}{2}} + ab + a^{\frac{2}{3}}b^{\frac{3}{2}} + a^{\frac{1}{3}}b^2 + b^{\frac{5}{2}})}{b^3-a^2}$.
30. $\dfrac{(x^{\frac{3}{2}} - xy^{\frac{1}{4}} + x^{\frac{1}{2}}y^{\frac{1}{2}} - y^{\frac{3}{4}})\,x^2}{x^2-y}$.
31. $\dfrac{x^2(x^{\frac{3}{2}} + xy^{\frac{1}{4}} + x^{\frac{1}{2}}y^{\frac{1}{2}} + y^{\frac{3}{4}})}{x^2-y}$.
32. $\dfrac{x^{\frac{27}{5}} + x^{\frac{24}{5}}y^{\frac{1}{2}} + x^{\frac{21}{5}}y + x^{\frac{18}{5}}y^{\frac{3}{2}} + x^3y^2 + x^{\frac{12}{5}}y^{\frac{5}{2}} + x^{\frac{9}{5}}y^3}{x^6-y^5}$

$$+ \frac{x^{\frac{6}{5}}y^{\frac{7}{2}} + x^{\frac{3}{5}}y^4 + y^{\frac{9}{2}}}{x^6 - y^5},$$

33. $\dfrac{x^{\frac{28}{3}}-x^{\frac{26}{3}}y^{\frac{2}{5}}+x^{8}y^{\frac{4}{5}}-x^{\frac{22}{3}}y^{\frac{6}{5}}+x^{\frac{20}{3}}y^{\frac{8}{5}}-x^{6}y^{2}+x^{\frac{16}{3}}y^{\frac{12}{5}}-x^{\frac{14}{3}}y^{\frac{14}{5}}}{x^{10}+y^{6}}$

$+\dfrac{x^{4}y^{\frac{16}{5}}-x^{\frac{10}{3}}y^{\frac{18}{5}}+x^{\frac{8}{3}}y^{4}-x^{2}y^{\frac{22}{5}}+x^{\frac{4}{3}}y^{\frac{24}{5}}-x^{\frac{2}{3}}y^{\frac{26}{5}}+y^{\frac{28}{5}}}{x^{10}+y^{6}}.$

34. $\dfrac{x^{\frac{2}{3}}y+x^{\frac{1}{3}}y^{\frac{2}{3}}+y^{\frac{1}{3}}}{xy-1}.$

35. $\dfrac{a^{\frac{1}{2}}y^{2}-ay^{\frac{5}{3}}+a^{\frac{3}{2}}y^{\frac{4}{3}}-a^{2}y+a^{\frac{5}{2}}y^{\frac{2}{3}}-a^{3}y^{\frac{1}{3}}}{y^{2}-a^{3}}.$

Page 131, Art. 319.

1. Given.
2. $x = 12$.
3. $x = 25$.
4. $x = \dfrac{(a-b)^2}{2a-b}$.
5. $x = \left(\dfrac{ab}{a-b}\right)^2$.
6. $x = 3$.
7. $x = mc$.
8. $x = \infty$, or ± 5.
9. $x = \frac{25}{16}$.

Page 135, Art. 330.

1. $x^2 - 5x - 14 = 0$.
2. $x^2 - (a+b)x + ab = 0$.
3. $x^2 - (a-b)x - ab = 0$.
4. $x^2 + (a+b)x + ab = 0$.
5. $x^2 - 6x + 12 = 0$.
6. $x^2 + 4x + 5 = 0$.

Page 137, Art. 335.

1. $x = \pm\frac{8}{3}$.
2. $x = \pm 3$.
3. $x = \frac{1}{2}[a-b\pm(a+b)\sqrt{-1}]$.
4. $x = 3$, or $-\frac{4}{5}$.
5. $x = 2 \pm \sqrt{-23}$.
6. $x = 5$, or $\frac{3}{2}$.
7. $x = \pm\frac{1}{2}\sqrt{2}$.
8. $x = \frac{1}{3}(a+b+c\pm\sqrt{a^2+b^2+c^2-ab-ac-bc})$.
9. $x = \pm 5$.
10. $x = 1$, or -2.
11. $x = \pm\sqrt{-bc} = -\frac{1}{2}\sqrt{bc}$, or $-\frac{3}{2}\sqrt{bc}$.
12. $x = \frac{1}{2}(1\pm\sqrt{17})$.
13. $x = \frac{1}{2}(a\pm\sqrt{a^2+b})$.
14. $x = 3$, or -4.
15. $x = \frac{3}{2}$, or $-\frac{5}{6}$.
16. $x = \frac{1}{2}(9\pm\sqrt{145})$.
17. $x = \dfrac{b\pm a}{c}$.
18. $x = \dfrac{n}{n^2-m^2}[bn \pm\sqrt{a^2(m^2-n^2)+b^2m^2}]$.
19. $x = \sqrt{n}\left(1\pm\dfrac{a}{\sqrt{m}}\right)$.

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PROBLEMS.

1. 15 and 6,
   or 35 and -14.
2. 10 and 4.
3. 17 and 9.
4. $\sqrt{\dfrac{a^2 \pm b^2}{2}}$.
5. 36.
6. 58 and 37.
7. 36 and 64.
8. 4 miles per hour.
9. 10.
10. $\dfrac{a}{2} \pm \dfrac{s}{2}\sqrt{2a - s^2}$.
11. 25 miles from O.
12. $\dfrac{n}{2}[n - a \pm \sqrt{(a-n)^2 - 4b}]$.

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13. 5000 in 1840,
    4000 in 1850,
    5200 in 1860,
    5300 in 1870.
14. 12 and 18.
15. 144.
16. 3 inches.
17. The man 36 yrs., son 16.
18. $\dfrac{\sqrt{m}}{2}\left(\dfrac{\sqrt{4n-1}+\sqrt{3}}{\sqrt{n-1}}\right)$, and
    $\dfrac{\sqrt{m}}{2}\left(\dfrac{\sqrt{4n-1}-\sqrt{3}}{\sqrt{n-1}}\right)$
19. 7.
20. 75 at \$4, or -120 at $-\$2.50$.
21. 256 sq. yds.

Page 141, Art. 339.

1. $x = (\frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4a})^n$.
2. $x = (-\frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 4b})^{2n}$.
3. $x = \pm\sqrt{a^2 - [-\frac{1}{4} \pm \frac{1}{4}\sqrt{1 + 8(m+n)}]^2}$.
4. $x = a - \dfrac{b^4}{2} - 2b^3 + 2b - 1 \pm (b^4 + 2b^3 - 2b^2)\sqrt{\dfrac{1}{b} + \dfrac{1}{4}}$.
5. $x = 27$, or -64.
6. $x = 64$, or 729.
7. $x = -\dfrac{\frac{1}{2} \pm \sqrt{n(n-1) + \frac{1}{4}}}{\frac{3}{2} \pm \sqrt{n(n-1) + \frac{1}{4}}}$.

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8. $x = 4$, or -1.
9. $x = \pm \dfrac{2a}{\sqrt{3}}$.

10. $x = 0$, or $-\frac{1}{2}(b \pm \sqrt{b^2 - 4a})$.
11. $x = 5$, or $26\frac{2}{3}$.
12. $x = \pm \frac{1}{2}\sqrt{3}$.
13. $x = 3$, or $-\frac{3}{4}$.
14. $x = \frac{a}{729}(1 \pm 2\sqrt{-2})^6$, or 0.
15. $x = 0$, 1 or 2.
16. $x = \pm 1$.
17. $x = -1$, or $\frac{1}{2}(-1 \pm \sqrt{-3})$.
18. $x = 1$, -1, $\frac{1}{2}(-1 \pm \sqrt{-3})$, or $\frac{1}{2}(1 \pm \sqrt{-3})$.
19. $x = \pm 3a$.
20. $x = 2$.
21. $x = \frac{a(1 \pm n)^2}{1 \pm 2n}$.
22. $x = \frac{2ab}{b^2 + 1}$.
23. $x = \pm \frac{1}{2}$.

Page 148, Art. 350.

5. $x = 5$; $y = 1$.
6. $x = 7$, or 2; $y = 2$, or 7.
7. $x = \frac{1}{2}\left(a \pm \sqrt{\frac{4b - a^3}{3a}}\right)$; $y = \frac{1}{2}\left(a \mp \sqrt{\frac{4b - a^3}{3a}}\right)$.
8. $x = 3$, or 2; $y = 2$, or 3.
9. $x = \pm 2$, or $\mp \frac{16}{5}\sqrt{5}$; $y = \pm 3$, or $\pm \frac{4}{5}\sqrt{5}$.
10. $x = 2$, or $\frac{1}{2}$; $y = 3$, or -24.
11. $x = 4$, 16, or $14 \pm \sqrt{58}$;
    $y = 5$, -7, or $-1 \pm \sqrt{58}$.
12. $x = \frac{1}{2}(1 \pm \sqrt{-7})$; $y = \frac{1}{2}(1 \mp \sqrt{-7})$.
13. $x = 4$, or -2; $y = 2$, or -4.
14. $x = \pm 3$; $y = \pm 1$.
15. $x = \pm 2$; $y = \pm 1$.

16. $x = \frac{1}{2}[a \pm \sqrt{a^2 + b} \pm \sqrt{b - 2(a^2 \pm a\sqrt{a^2 + b})}]$;

$y = \frac{1}{2}[a \pm \sqrt{a^2 + b} \mp \sqrt{b - 2(a^2 \pm a\sqrt{a^2 + b})}]$.

17. $x = 1$, or -2; $y = -2$, or 1.

18. $x = 16$, or 4; $y = 4$, or 16.

19. $x = 9$, or 1; $y = 1$, or 9.

20. $x = 2$; $y = 1$.

21. $x = 4$; $y = 2$.

22. $x = 2$, or 4; $y = 4$, or 2.

23. $x = 5$; $y = 3$.

24. $x = \pm 6$, or ± 3; $y = \pm 3$, or ± 6.

25. $x = \pm 5$; $y = \pm 1$.

26. $x = 2$; $y = 1$.

27. $x = 4$, or 32; $y = -3$, or $\frac{1}{2}$.

28. $x = 3, 2$, or $-2 \pm \frac{1}{2}\sqrt{10}$;

$y = 2, 3$, or $-2 \mp \frac{1}{2}\sqrt{10}$.

29. $x = 5$, or $\frac{5}{4}$; $y = 2$, or $\frac{1}{2}$.

30. $x = 2$; $y = 3$.

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PROBLEMS.

2. 7, 11, and 23.
3. 6, 13, and 25.
4. 4 and 5 yards.
5. 36.

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6. 15, 12, and 9.
7. A's, \$80 at 5%; B's, \$120 at 6%.
8. A's, \$100; B's, \$150.
9. $\frac{1}{b^2 - a^2}(-an^2 \pm b\sqrt{b^2m^2 - a^2m^2 + n^4})$, and

$\frac{1}{b^2 - a^2}(-bn^2 \pm a\sqrt{b^2m^2 - a^2m^2 + n^4})$.

10. The parts of a are

$$\frac{1}{2b}\left[ab - m + n \pm \sqrt{(ab - m - n)^2 - 4mn}\right], \text{ and}$$

$$\frac{1}{2b}\left[ab + m - n \mp \sqrt{(ab - m - n)^2 - 4mn}\right].$$

The parts of b are

$$\frac{1}{2a}\left[ab + m - n \pm \sqrt{(ab - m - n)^2 - 4mn}\right], \text{ and}$$

$$\frac{1}{2a}\left[ab - m + n \mp \sqrt{(ab - m - n)^2 - 4mn}\right].$$

11. $\sqrt[4]{3}$.

12. $\frac{1}{2}(3 + \sqrt{-3})$ and $\frac{1}{2}(3 - \sqrt{-3})$.

13. $\frac{1}{2}(3 \pm \sqrt{5})$ and $\frac{1}{2}(1 \pm \sqrt{5})$.

14. $\dfrac{5 \pm \sqrt{5}}{6 \pm 2\sqrt{5}}$ and $-\dfrac{5 \pm 3\sqrt{5}}{6 \pm 2\sqrt{5}}$.

15. ± 3 and ± 1.

16. A had 200 acres, at \$1.50 per acre;
B had 400 acres, at \$.75 per acre.

Page 153, Art. 357.

1, 2. Given.

3. $x > 4$, the limit of x.

4. $x > \frac{24}{5}$, the limit of x.

5. $x > a$, $x < b$, the limits of x.

6. $x < 5$, $x > 3$; $\therefore$ 4 is the only integral value of x.

7. $x < 6$ and $x > 4$; $\therefore$ 3, 4, and 5, are the integral values of x.

8. The number is 19 or 20.

9. He sold 60 apples.

10. 20 and 5.

Page 160.

PROBLEMS.

1. The third proportional is 100.

2. The first term is 54.

3. The mean proportional is 15.

4. In $\dfrac{abc}{a'b'}$ days.

5. 16 and 20 tons.

6. A, $54\frac{3}{5}$ hours at $9\frac{1}{19}$ miles per hour $= 494\frac{26}{95}$ miles;
B, $59\frac{3}{5}$ hours at $11\frac{1}{19}$ miles per hour $= 570\frac{6}{19}$ miles;
A, 3 hours at 0 miles per hour $=$ 0 miles;
B, 0 hours at 2 miles per hour $=$ 0 miles.

7. The numbers are 30, 48, 50.

8. 6, or $3\,(-1 \pm \sqrt{-3})$; and 9, or $\frac{9}{2}\,(-1 \pm \sqrt{-3})$.

9. The number is 863.

10. The numbers are a, $2a$, and $3a$.

11. The conditions are not independent.

Page 162, Art. 381.

1. Given.

2. $x = \dfrac{m}{y^2}$; $xy^2 = m$; $x_1 : x_2 = y_2{}^2 : y_1{}^2$.

3. $x = m\,(a + y)$; $x_1 : x_2 = a + y_1 : a + y_2$.

4. $x = m\,(y^2 + y^3)$; $x_1 : x_2 = y_1{}^2 + y_1{}^3 : y_2{}^2 + y_2{}^3$.

5. $x = \dfrac{m}{y + y^2}$; $x_1 : x_2 = y_2 + y_2{}^2 : y_1 + y_1{}^2$.

6. The values of y are 25, $11\frac{2}{3}$; of x, 12, $4\frac{4}{5}$.

7. The values of x are $\frac{49}{25}$, $\frac{9}{25}$, $\frac{4}{25}$, 0.

8. The value of m is 1.

9. The value of y is $\frac{3}{2}\sqrt[3]{2x^2}$.

10. $y = 2\left(x + \dfrac{1}{x}\right)$.

11. 19200 inches. 5 seconds.

Page 165, Art. 388.

1. 362880.
2. 15120.
3. 126.
4. 126.

5. $C_{\frac{m}{n}} = \dfrac{n(n-1)\ .\ .\ .\ .\ .\ (n-m+1)}{\lfloor m} =$

$\dfrac{n(n-1)\ldots\ldots(n-m+1)\ \lfloor n-m}{\lfloor n-m\ \lfloor m} = \dfrac{\lfloor n}{\lfloor n-m\ \lfloor m};$

$C_{\frac{n-m}{n}} = \dfrac{n(n-1)\ldots\ldots[n-(n-m)+1]}{\lfloor n-m} =$

$\dfrac{n(n-1)\ldots(m+1)\ \lfloor m}{\lfloor n-m\ \lfloor m} = \dfrac{\lfloor n}{\lfloor n-m\ \lfloor m}.$

6. $C_{\frac{2}{6}} = 15.$
7. $C_{\frac{3}{9}} = 84.$
8. No more.
9. $\lfloor m.$
10. Zero.
11. 8, or -1. The 2d Ans. not applicable.
12. $m = n$.
13. $m = n$, or $n + 1$.
15. 454053600.
16. 3991680.

Page 175, Art. 417.

1–6. Given.

7. $\dfrac{dy}{dx} = (x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b).$

8. $\dfrac{dy}{dx} = \dfrac{x^2 + 2x - 1}{(x+1)^2}.$

9. $\dfrac{dy}{dx} = \dfrac{1 - x^2}{(x^2+1)^2}.$

10. $\dfrac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} - \dfrac{6}{x^{\frac{1}{5}}} - \dfrac{2}{x^{\frac{3}{5}}};\ \dfrac{d^2y}{dx^2} = \dfrac{3}{4x^{\frac{1}{2}}} + \dfrac{2}{x^{\frac{4}{3}}} + \dfrac{6}{5x^{\frac{6}{5}}}.$

11. $\dfrac{dy}{dx} = \dfrac{1}{(1-x)^2}.$

12. $\dfrac{dy}{dx} = B + 2Cx + 3Dx^2.$

13. $\dfrac{dy}{dx} = 5(1+x)^4.$

14. $\dfrac{dy}{dx} = -4(1-x)^3.$

15. $\frac{dy}{dx} = n(1 + x)^{n-1}$

16. $\frac{dy}{dx} = -m(1 - x)^{m-1}.$

17. $\frac{dy}{dx} = 7x^6 - 20x^3 + 6x.$

18. $\frac{dy}{dx} = (x-1)(x+2)(x-5) + (x-1)(x+2)(x+3)$
$+ (x-1)(x-5)(x+3) + (x+2)(x-5)(x+3).$

19. Given.

20. $\frac{dy}{dx} = \frac{\sqrt{a}\,(y - x)}{2\sqrt{b}\,\sqrt{xy^3}}.$

21. $\frac{dx}{dy} = -\frac{1}{2(1 + x)^{\frac{3}{2}}}.$

22. $\frac{dy}{dx} = \frac{4x}{(1 - x^2)^{-3}}.$

23. $\frac{dy}{dx} = \frac{2x}{3(a + x^2)^{\frac{2}{3}}}.$

24. $du = 2xy^3dx + 3x^2y^2dy + 3x^2y^2dx + 2x^3ydy.$

25. $\frac{du}{dx} = x^2 - x.$

Page 179, Art. 422.

2. Given.

3. $x - x^2 + x^3 - x^4 +$ etc.

4. $\frac{1}{x} - 1 + x - x^2 +$ etc.

5. $x^{\frac{1}{2}} - \frac{1}{2x^{\frac{1}{2}}} - \frac{1}{2\cdot 4x^{\frac{3}{2}}} - \frac{1}{2\cdot 8x^{\frac{5}{2}}} -$ etc.

6. $\frac{a}{x^2} + \frac{a - 1}{x} + (a - 1)(1 + x + x^2 +$ etc.$)$

7. $\frac{1}{a^{\frac{1}{2}}} - \frac{x}{2a^{\frac{3}{2}}} + \frac{3x^2}{2\cdot 4a^{\frac{5}{2}}} -$ etc

8. $\frac{1}{a} - \frac{2}{a^2}x + \frac{3}{a^3}x^2 - \frac{4}{a^4}x^3 +$ etc.

9. $-1 - 2x - 2x^2 - x^3 + 2x^4 + 7x^5 +$ etc.

10. $1 + x + x^2 + x^3 + x^4 +$ etc.

11. $1 - 3x + 3x^2 - 3x^4 + 3x^5 -$ etc.

12. $\frac{1}{a} - \frac{x}{a^2} + \frac{x^2}{a^3} - \frac{x^3}{a^4} +$ etc.

13. $\frac{1}{x} - \frac{1}{x^2}$.

14. $1 + \frac{x}{a} + \frac{x^2}{a^2} + \frac{x^3}{a^3} +$ etc.

15. $x^{\frac{1}{3}} - \frac{1}{3x^{\frac{2}{3}}} - \frac{1}{9x^{\frac{5}{3}}} - \frac{5}{81x^{\frac{8}{3}}} -$ etc.

16. $-\frac{1}{\sqrt{x}}$.

17. $1 + x^{\frac{1}{3}} + x + x^{\frac{5}{3}} - x^{\frac{7}{3}} + x^4 -$ etc.

18. $-x^{\frac{1}{3}} + 3x^{\frac{2}{3}} - 5x + 7x^{\frac{4}{3}} - 9x^{\frac{5}{3}} +$ etc.

19. $1 + x^{\frac{1}{4}} + x^{\frac{1}{2}} + x^{\frac{3}{4}} + x + x^{\frac{5}{4}} +$ etc.

20. $x^{\frac{1}{2}} + x^{\frac{3}{2}} + x^{\frac{5}{2}} + x^{\frac{7}{2}} +$ etc.

Page 183, Art. 430.

1, 2. Given.

3. $\frac{3}{14(x-2)} + \frac{31}{35(x+5)} - \frac{1}{10x}$.

4. $\frac{1}{3(x+1)^2} + \frac{5}{3(x+1)} - \frac{1}{x}$.

5. $\frac{1}{2(x-1)} + \frac{x+2}{2(x^2+x+4)}$.

6. $\frac{1}{49x} + \frac{672-105x}{49(x^2+2x+7)^2} + \frac{48x-100}{49(x^2+2x+7)}$.

7. $\frac{113}{40(x-5)} - \frac{1}{5x} - \frac{25}{40(x+3)} + 1$.

8. $\frac{a^2}{x-1} + \frac{a-a^2}{(x-1)^2} - \frac{a^2}{x}$.

9. $\frac{1}{4(x-1)} - \frac{1}{4(x+1)} - \frac{1}{2(x^2+1)}$.

10. $\frac{1}{9(x-2)^2} - \frac{2}{27(x-2)} + \frac{2}{27(x+1)} + \frac{1}{9(x+1)^2}$.

11. $\dfrac{1}{(a+b)(x-b)} - \dfrac{1}{(a+b)(x+a)}$.

12. $\dfrac{20}{9(x+2)} + \dfrac{1}{x-2} + \dfrac{7}{4(x-2)^2} + \dfrac{23}{12(x+2)^2}$
$+ \dfrac{5}{18(x-1)} - \dfrac{3}{2(x+1)}$.

Page 186, Art. 433.

1. $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$.

2. $a^{\frac{1}{2}} - \dfrac{b}{2a^{\frac{1}{2}}} - \dfrac{b^2}{8a^{\frac{3}{2}}} - \dfrac{b^3}{16a^{\frac{5}{2}}} -$ etc.

3. $\dfrac{1}{a^3} + \dfrac{3b}{a^4} + \dfrac{6b^2}{a^5} + \dfrac{10b^3}{a^6} + \dfrac{15b^4}{a^7} +$ etc.

4. $\dfrac{1}{a} - \dfrac{b}{a^2} + \dfrac{b^2}{a^3} - \dfrac{b^3}{a^4} +$ etc.

5. $\dfrac{1}{x^{\frac{1}{2}}} + \dfrac{y}{2x^{\frac{3}{2}}} + \dfrac{3y^2}{8x^{\frac{5}{2}}} + \dfrac{5y^3}{16x^{\frac{7}{2}}} +$ etc.

6. $\dfrac{1}{x^2} + \dfrac{2y}{x^3} + \dfrac{3y^2}{x^4} + \dfrac{4y^3}{x^5} +$ etc.

7. $m^{\frac{2}{3}} + \dfrac{2n}{3m^{\frac{1}{3}}} - \dfrac{n^2}{9m^{\frac{4}{3}}} + \dfrac{4n^3}{81m^{\frac{7}{3}}} - \dfrac{7n^4}{243m^{\frac{10}{3}}} +$ etc.

8. $\dfrac{1}{m^{\frac{2}{3}}} + \dfrac{2n}{3m^{\frac{5}{3}}} + \dfrac{5n^2}{9m^{\frac{8}{3}}} + \dfrac{40n^3}{81m^{\frac{11}{3}}} +$ etc.

9. $\dfrac{1}{m^{\frac{2}{3}}} - \dfrac{2n}{3m^{\frac{5}{3}}} + \dfrac{5n^2}{9m^{\frac{8}{3}}} - \dfrac{40n^3}{81m^{\frac{11}{3}}} +$ etc.

10. $x^{\frac{2}{3}} - \dfrac{4}{3x^{\frac{1}{3}}} - \dfrac{4}{9x^{\frac{4}{3}}} - \dfrac{32}{81x^{\frac{7}{3}}} -$ etc.

11. $x^3 + 6x^2 + 12x + 8$.

12. $x^{\frac{1}{2}} + \dfrac{1}{x^{\frac{1}{2}}} - \dfrac{1}{2x^{\frac{3}{2}}} + \dfrac{1}{2x^{\frac{5}{2}}} - \dfrac{5}{8x^{\frac{7}{2}}} +$ etc.

13. $a^5x^5 + 5a^4bx^4y + 10a^3b^2x^3y^2 + 10a^2b^3x^2y^3 + 5ab^4xy^4 + b^5y^5$.

14. $a^{\frac{1}{2}}x^{\frac{1}{2}} + \dfrac{by}{2a^{\frac{1}{2}}x^{\frac{1}{2}}} - \dfrac{b^2y^2}{8a^{\frac{3}{2}}x^{\frac{3}{2}}} + \dfrac{b^3y^3}{16a^{\frac{5}{2}}x^{\frac{5}{2}}} -$ etc.

15. $\dfrac{1}{a^{\frac{1}{2}}x^{\frac{1}{2}}} - \dfrac{by}{2a^{\frac{3}{2}}x^{\frac{3}{2}}} + \dfrac{3b^2y^2}{8a^{\frac{5}{2}}x^{\frac{5}{2}}} - \dfrac{5b^3y^3}{16a^{\frac{7}{2}}x^{\frac{7}{2}}} +$ etc.

16. $25a^2 - 70ax + 49x^2$.

17. $3^{\frac{1}{2}}a^{\frac{1}{2}} + \dfrac{2x}{3^{\frac{1}{2}}a^{\frac{1}{2}}} - \dfrac{2x^2}{3^{\frac{3}{2}}a^{\frac{3}{2}}} + \dfrac{4x^3}{3^{\frac{5}{2}}a^{\frac{5}{2}}} -$ etc.

18. $-^{n}\dfrac{1 \cdot 3 \cdot 5 \ldots\ldots (2n-5)}{2^{n-1} \cdot 1 \cdot 2 \cdot 3 \cdot (n-1)}a^{\frac{1}{2}-n}x^{\frac{1}{2}-n}b^{n-1}$.

19. $-^{m-1}m^2x^{m-1}$.

20. $\dfrac{1 \cdot 3 \cdot 5 \ldots\ldots (2n-3)}{2^{n-1} \cdot 1 \cdot 2 \cdot 3 \cdot (n-1)}a^{\frac{1}{2}-n}x^{n-1}$.

Page 192, Art. 448.

1–3. Given.	4. 0.	5. 1.	6. − 5.
7. − 3.	8. − 2.	9. − 1.	10. 0.
11. 2.	12. 4.	13. 6.	

Page 196, Art. 459.

2.	1548.781.	8.	14.385416+.
3	1.973355.	9.	2.7234+.
4.	− 111.	10.	2906.25.
6.	78.	11.	.1813+.
7.	.0375+.	12.	− 4.619+.

Page 197, Art. 461.

13.	Given.	19.	1.0837+.
14.	.00033215.	20.	2.5047+.
15.	33.3336+.	21.	2.1248+.
16.	191.8139.	23.	0.34284+.
17.	1.3.	24.	0.54613+.
18.	5.23176+.	25.	0.32471+.

Page 215, Art. 497.

1. $a_n = 3n - 2$; $S_n = \dfrac{3n^2 - n}{2}$

2. $a_n = 3\frac{1}{2}n - \frac{1}{2}$; $S_n = \dfrac{7n^2 + 5n}{4}$.

3. $a_n = 5n - 3$; $S_n = \dfrac{5n^2 - n}{2}$.

4. $a_n = \dfrac{3n^2 - 7^n + 6}{2}$; $S_n = \dfrac{n(n^2 - 2n + 3)}{2}$.

5. $a_n = n$; $S_n = \dfrac{n(n+1)}{2}$.

6. $a_n = 2n - 1$; $S_n = n^2$.

7. $a_n = 2n$; $S_n = n + n^2$.

8. $a_n = \dfrac{n(n+1)}{2}$; $S_n = \dfrac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3}$.

9. $a_n = n^2$; $S_n = \dfrac{n(n+1)(2n+1)}{1 \cdot 2 \cdot 3}$.

10. $a_n = n^3$; $S_n = \dfrac{n^2(n+1)^2}{4}$.

11. $a_n = \dfrac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3}$; $S_n = \dfrac{n(n+1)(n+2)(n+3)}{1 \cdot 2 \cdot 3 \cdot 4}$.

12. $a_n = \dfrac{n(n+1)(n+2)(n+3)}{1 \cdot 2 \cdot 3 \cdot 4}$;

$S_n = \dfrac{n(n+1)(n+2)(n+3)(n+4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$.

13. $a_n = \dfrac{n(n+1)(n+2)(n+3)(n+4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$;

$S_n = \dfrac{n(n+1)(n+2)(n+3)(n+4)(n+5)}{\lfloor\underline{6}}$.

14. (1) 0. (2) 0. (3) 0.

15. (1) The eighth. (2) The ninth.

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16. 220 balls in a triangular pyramid.

17. 385 balls in a quadrangular pyramid.

18. 280 balls in an oblong rectangular pile.

19. The $-$ 10th term is 210.

20. (1) 304 ft. (2) 1600 ft. (3) 900 ft. (4) $\frac{4096}{9}$ ft. (5) $16n^2$ ft. 2,4

Page 223, Art. 513.

1. $a_{20} = 524288,$ $\quad S_{20} = 1048575;$
   $a_n = 2^{n-1};$ $\quad S_n = 2^n - 1.$
2. $a_{10} = 59049,$ $\quad S_{10} = 88572;$
   $a_n = 3^n;$ $\quad S_n = \frac{3}{2}(3^n - 1).$
3. $a_{10} = 2560,$ $\quad S_{10} = 5115;$
   $a_n = 5 \cdot 2^{n-1};$ $\quad S_n = 5(2^n - 1).$

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4. $a_\infty = 0,$ $\quad S_\infty = 16.$
5. $a_\infty = 0,$ $\quad S_\infty = 5\frac{1}{3}.$
6. $m' = \frac{1}{2},$ $\quad a_{5\frac{1}{2}} = \frac{1}{128}.$
7. $m' = 2,$ $\quad a_{1\frac{1}{3}} = 2,$ $\quad a_{1\frac{2}{3}} = 4,$
   $a_{2\frac{1}{3}} = 16,$ $\quad a_{2\frac{2}{3}} = 32.$
8. $a_{10} = -19683,$ $\quad S_{10} = -14762;$
   $a_5 = 81;$ $\quad S_5 = 61.$
9. The scale is $2x^3, \; -3x^2, \; 3x.$
   Series continued is $86x^7, \; 171x^8, \; 341x^9, \; 672x^{10}$, etc.
10. The scale is $-x^2, \; 2x.$ $\quad a_{12} = 35x''.$

Page 227, Art. 520.

1. The fifth term is $\frac{1}{10}$.
2. $\frac{1}{6}, \frac{1}{10}, \frac{1}{14}, \frac{1}{18}.$
3. $5, \; 6\frac{1}{4}, \; 8\frac{1}{3}, \; 12\frac{1}{2}, \; 25.$
4. See Key.

Page 227, Art. 521.

1. $a = p(1 + r)^t.$
2. $p = \dfrac{rs(1 + r)^t}{(1 + r)^t - 1}.$
3. $s = \dfrac{a}{(1 + r)^t}.$

Page 228.

4. $w = \dfrac{s}{(1 + r)^t}$.

5. $w = \dfrac{a}{r(1 + r)^t}$.

6. $w = \dfrac{a[(1 + r)^{t'} - 1]}{r(1 + r)^{t+t'}}$.

7. \$595.58.

8. \$871.73.

9. \$783.53.

10. $w = \dfrac{a[(1 + r)^{10} - 1]}{r(1 + r)^{12}}$.

11. $w = \dfrac{a[(1 + r)^7 - 1]}{r(1 + r)^4}$

12. \$1245.43.

13. \$1666.66.

14. \$2230.38.

15. April 27th, 1875.

16. Answers in order in miles 160, 55, 55, 10, -11, -270, -4.

17. 40.951 miles; 61.25+ miles; 2.73+ days; 100 miles.

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18. ∞, 300 ft.

Page 230, Art. 522.

4. $\frac{5}{24}$.
5. $\frac{1}{12}$.
6. $\frac{3}{4}$.
7. $\frac{1}{4}$.
8. 1.
9. $\frac{1}{18}$.
10. $\frac{1}{90}$.
11. $\frac{7}{36}$.
12. $\frac{11}{90}$.

Page 231, Art. 523.

1. $x = y - 2y^2 + 5y^3 - 9y^4 +$ etc.

2. $x = 1 - y + (1 - y)^2 + (1 - y)^3 + (1 - y)^4 +$ etc.

3. $x = y + \frac{1}{2}y^2 + \frac{1}{6}y^3 + \frac{1}{24}y^4 + \frac{1}{120}y^5 +$ etc.

4. $x = y - 1 - 2(y - 1)^2 + 5(y - 1)^3 - 9(y - 1)^4 +$ etc.

Page 243, Art. 542.

2. $z^7 - 15z^5 + 848z^2 - 243z - 729 = 0.$

3. $z^4 - 3z^3 + 45z^2 + 125 = 0.$

4. $z^4 - 3z^3 - 7z^2 + 8 = 0.$

Page 244, Art. 545.

1. $-1,\quad 4,$ and $5.$

2. $1,\quad -1,\quad 2,\quad 3,\quad 6.$

3. $1.$

4. $2,\quad -2.$

5. $5.$

6. None.

Page 246, Art. 548.

1. $x^6 - 2x^5 - 37x^4 + 68x^3 + 327x^2 - 450x - 675 = 0.$

2. $$x^4 \begin{array}{l|l|l|l} -a & x^3 + ab & x^2 - abc & x + abcd = 0. \\ -b & \quad + ac & \quad - abd & \\ -c & \quad + ad & \quad - acd & \\ -d & \quad + bc & \quad - bcd & \\ & \quad + bd & & \\ & \quad + cd & & \end{array}$$

3. $x^3 - 8 = 0.$

4. $x^4 + 6x^3 + 7x^2 - 24x - 44 = 0.$

5. $x^6 - 8x^5 + 31x^4 - 40x^3 + 6x^2 + 288 = 0.$

Page 251, Art. 553.

1. 4 positive, 1 negative, 2 imaginary.

2. 2 positive, 2 negative, 2 imaginary.

3. 0 positive, 1 negative, 4 imaginary.

4. 1 positive, 0 negative, 4 imaginary.

Page 262, Art. 567.

1–6. Not desirable to give.

7. $z^4 - 6z^3 + 28z^2 + z - 16 = 0.\qquad z = 2x^{\frac{1}{2}},$

8. $z^5 - 6z^3 + 28z^2 - 48z - 32 = 0.$ $z = 2x^{\frac{1}{5}}.$

9. $z^{12} - 64z^6 - 384z^4 - 512z^3 - 4096 = 0.$ $z = 2x^{\frac{1}{12}}.$

10. $z^5 + 2z^4 + 243z^2 - 13122 = 0.$ $z = 9x^{\frac{1}{5}}.$

11. $z^2 + z - 6 = 0.$ $z = 2x.$

12. $x + 1 = 0.$

13. All the roots are included in the equal roots, and the result of removing them is $1 = 1$, an equation of the zero degree and having no roots.

14. The same as the last.

The equations are as follows:

1. $x^3 - 2x^2 - 5x + 6 = 0.$

2. $x^4 - 7x^3 + 10x^2 + 14x - 24 = 0.$

3. $x^3 + 5x^2 + 2x + 10 = 0.$

4. $2x^4 - x^3 - 16x + 8 = 0.$

5. $x^4 - 8x^3 + 27x^2 - 46x + 44 = 0.$

6. $x^6 - 12x^5 + 58x^4 - 144x^3 + 193x^2 - 132x + 36 = 0.$

7. $x^4 + 2\frac{7}{8}x^3 + \frac{110}{35}x^2 - \frac{409}{144}x + \frac{35}{144} = 0.$

8. $6x^5 - 41x^4 + 97x^3 - 97x^2 + 41x - 6 = 0.$

9. $x^4 - 4 = 0.$

10. $x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32 = 0.$

Page 266, Art. 575.

2. $x = -3.8+.$

3. $x = 6.5+.$

4. $x = 0.3+.$

5. $x = -1.0+,$ $1.3+,$ and $4.6+.$

6. $x = -1.3+.$

7. $x = \pm 1.4+$ and $\pm 1.7+.$

8. $x = 1.4+,$ $5.2+,$ and $-0.6+.$

9. $x = -6.6+.$

10. $x = 0.4+,$ $0.7+,$ and $-6.2+.$

Page 271, Arts. 578, 579.

2. $x = 1.3797+$.
3. $x = 125$.
4. $x = 0.601+$ and $-1.6+$.
5. $x = 8.8+$.
6. $x = 1.259921+$.
7. $x = 0.90+$.
8. $x = 1.3+$, $1.6+$, and $-3.04+$.
9. $x = 2.6+$.
10. $x = -2.5+$.

Page 277, Art. 589.

1. Given.
2. $x = 1$, or $\frac{1}{2}(1 \pm \sqrt{-3})$.
3. $x = \frac{1}{4}(3 \pm \sqrt{-7} \pm \sqrt{-62 \mp 6\sqrt{-7}})$.
4. $x = \pm 1$, or $\frac{1}{2}(-7 \pm 3\sqrt{5})$.
5. $x = 1$, or $\frac{1}{12}(-1 \pm \sqrt{71} \pm \sqrt{-72 \mp \sqrt{71}})$.
6. $x = \pm 1$, or $\frac{1}{4}(3 \pm \sqrt{15} \pm \sqrt{8 \mp \sqrt{15}})$.
7. $x = \frac{1}{4}(1 \pm \sqrt{-3} \pm \sqrt{-18 \mp 2\sqrt{-3}})$.
8. $x = \pm 1$, or $\frac{1}{2}(1 \pm \sqrt{-3})$.
9. $x = \pm 1$, or $\dfrac{1}{a}(1 \pm \sqrt{1 - a^2})$.
10. $x = -\frac{1}{10}(-4 \pm \sqrt{-29} \mp \sqrt{-113 \mp 8\sqrt{-29}})$.

Page 278, Art. 592.

1. Given.
2. $x = \pm 1$, $-\frac{1}{2} \pm \frac{1}{2}\sqrt{-3}$, and $\frac{1}{2} \pm \frac{1}{2}\sqrt{-3}$.
3. $x = \pm 1$, or $\pm\sqrt{-1}$, for $x^4 - 1 = 0$;

   $x = \pm\sqrt{\pm\sqrt{-1}}$, for $x^4 + 1 = 0$.

4. $x = \pm 1$, or $\pm \frac{1}{2}(1 \pm \sqrt{-3})$, for $x^6 - 1 = 0$;

$x = \pm \sqrt{-1}$, or $\pm \sqrt{\frac{1}{2}(1 \pm \sqrt{-3})}$, for $x^6 + 1 = 0$.

5. See Example 1 for $x^5 - 1 = 0$.

$x = -1$, or $\frac{1}{4}(1 \pm \sqrt{5} \pm \sqrt{-10 \pm 2\sqrt{5}})$, for $x^5 + 1 = 0$.

6. Multiply the roots of $x^5 + 1 = 0$ by $7^{\frac{1}{5}}$.

7. Multiply the roots of $x^5 - 1 = 0$ by $4^{\frac{1}{5}}$.

8. Multiply the roots of $x^6 + 1 = 0$ by $5^{\frac{1}{6}}$.

9. Multiply the roots of $x^6 - 1 = 0$ by $3^{\frac{1}{6}}$.

Page 279, Art. 593.

1–2. Given.

3. $x = 1.7558749+$.

4. $x = 1.3+$.

5. $x = 1.9+$.

6. $x = 1.6+$.

7. In $12+$ years.

8. In $\dfrac{\log A - \log a}{\log(1 + r)} + 1$ years.

9. In $26.75+$ years.

Page 281, Art. 596.

1–2. Given.

3. $\frac{7}{12}$.

4. $\frac{1}{970200}$.

5. $\frac{3}{4}$.

6. $\frac{1}{4}$ and $\frac{1}{8}$.

7. $\frac{5}{16}$.

8. $\frac{125}{1331}$.

9. $\frac{6}{77}$.

10. $\frac{5}{22}$.

Page 284, Art. 599.

1–3. Given.

4. $x = 2$, or $\frac{1}{2} \pm \frac{1}{2}\sqrt{19}$.

5. $x = 1$, or $-\frac{1}{2} \pm \frac{1}{2}\sqrt{23}$.

6. $x = 2$, or $-1 \pm 3\sqrt{-1}$.
7. $x = 4.07+$.
8. $x = 3.9+$, $-3.8+$, and $-0.13+$.
9. $x = 2.6+$, $0.58+$, and $-3.2+$.
10. $x = 1.8+$, $-1.5+$, and $-0.3+$.
11. $x = 2.1+$, $0.7+$, and $-2.9+$.
12. $x = 2.1+$, $0.5+$, and $-0.6+$.

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MISCELLANEOUS EXAMPLES.

1. $\dfrac{x-5}{x+5}$.
2. $\dfrac{26a^2 + 38ax}{35a^2 + 66ax + 27x^2}$.
3. $\dfrac{3a^3 + 9a^2x + 6ax^2 + 2x^3}{2a^4 + 3a^3x - a^2x^2 - 3ax^3 - x^4}$.
4. $x = 1$.
5. In 30 hours.
6. 5 hours 27 minutes 16.36 seconds.
7. $2x + 3y + z$.
8. $x + 2$.
9. $a^3 + 3a + 2$.
10. $x = \frac{1}{36}(-1 \pm \sqrt{649})$.
11. 30 gallons.

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12. 9.369 miles, nearly.
13. $1000.
14. The majority 804; the minority 603.
15. $x = b$.
16. $x = 2a$.

17. $\pm \frac{\sqrt{a}}{2}\left[\frac{a \pm \sqrt{a^2+2}}{(a^2+2)^{\frac{1}{2}}}\right]$.

18. $x = \pm \frac{1}{a}[\pm(\sqrt{1+a^2}-1)(\sqrt{1-a^2}+1)]^{\frac{1}{2}}$.

19. $x = \frac{1}{2} \pm \frac{1}{2}\sqrt{5}$.

20. In 40 minutes after they started.

21. Rate, 3, 5, and 4 miles an hour respectively.

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22. $y = \frac{a^4x^2}{b^4(a+x)}$.

23. 162.

24. $\dfrac{1 - \dfrac{1}{(1+\sqrt{2})^n}}{\sqrt{2}}$.

25. 100.

26. See Key.

27. Distance apart 450 miles. A's rate 30 miles and B's rate 25 miles per day.

28. The rate of increase was fourfold.

29. The first point of meeting was at the distance $\frac{d}{\sqrt{2}}$ from one starting point, and the second the same distance from the other, d being the distance between the two points. The ratio of their rates is $1 + \sqrt{2}$.

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30. He had 115 hurdles and they must be 1.7 ft. apart (nearly).

31. 12 and 8 gallons.

32. $x = 1, \quad -4, \quad$ and $\frac{3}{2}(1 \pm \sqrt{41})$.

33. $(2a - 1)(b^2 - 4c^2 - ab) = 0$.

34. The parts are 30, 36, and 45.

35, 36. See Key.

37. $\dfrac{a(1 - m^n)}{m^{n-1} - m^n}$.

38. 23.456+.

39–41. See Key.

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42. $x = 2, \quad -1.84+, \quad 3.13+,$ and $-3.28+$;
$y = 3, \quad 3.58+, \quad -2.805+,$ and $-3.77+$.

43. $x = -a \pm \sqrt{a^2 + b^2}$;

$$y = b \pm \frac{\sqrt{2a^2 + 2b^2 \mp \sqrt{a^2 + b^2}}}{2}.$$

44. See Key.

45. \$600000.

46. In 1845, Population 4354;
" 1854, " 5602;
" 1862, " 6943;
" 1880, " 11478.

47. $-\dfrac{l}{b - b'}[b' \pm \sqrt{\tfrac{1}{2}(b^2 + b'^2)}]$; l being the length and b and b' the width of the ends.

48. The first is $2x^2 + 4x$; the second is $2x^3 + 3x^2 + 0$.

49. In 41.06+ years.

50. The chance is $\frac{29}{30}$.